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Introduction to engineering fluid mechanics

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Introduction

to

Engineering Fluid Mechanics

Preface

In the accompanying material the authors endeavor to provide, primarily for the competent undergraduate in mechanical, chemical, marine, aeronautical or related engineering curricula, a Fluid Mechanics program in which -

(a) He may acquire a sufficient familiarity with the empirical and admittedly approximate methods which of necessity the engineer employs quite effectively to his many practical purposes; and

(b) He may then obtain, with better appreciation of their objectives, at least an introductory knowledge of the concepts and techniques required for subsequent consideration of kinematic aspects of the flow phenomena with which he deals, and of the flow patterns and pressure distributions assumed in a stream when in passage about restraining surfaces.

In giving a primary position to engineering manners of approach we reflect a general opinion that, through their ability to organize so effectively our experience concerning the passage of real fluids (rather then <u>ideal</u> ones), in complex flow configurations and through channels of perhaps difficult controllable nature of such approaches have an unasailable place in fluid-mechanics literature. But we endeavor to make evident both their limitations and such over-simplifications as are employed and are in fact welcomed, but which need be sufficiently recognized.

In delaying anything but incidental attention to kinematic aspects of flow we are guided by considerable experience indicating that the student recognizes much more readily the significance of such considerations after having first acquired a general view of the purposes and utility of fluid-mechanics study. But, as the undergraduate does not normally possess the mathematical tools required for the more exacting study, attention is for the most part limited to basic concepts and to such elementary flow patterns and their combinations as may be analyzed with his mathematical equipment.

In this connection we venture to express a regret that in the current literature of fluid mechanics, some as contributed by hydraulicists of more practical engineering background and some as developed more recently by aerodynamicists of mathematico-physical background, there is frequently such a diversity of viewpoint and objectives, and a failure to indicate their relationship or the limitations of either approach alone, that it may well be quite perplexing to the engineering student. We shall be relieved if this work may contribute to his acquisition of a better coordinated picture of the general science. The material has been used with gratifying results with numerous student groups of various technical background and in preparation for diverse subsequent responsibilities.

Several items which may be of further prefatory interest are as follows -

(a) Instead of emphasis on the concept of "head" which the earlier hydraulicist believed to be convenience, we find that direct attention to the fundamental concept of energy (per unit mass of fluid) offers many advantages. Also confusions may so be avoided when utilizing the principle of dimensional homogeneity when coordinating performances in situations where geometric and dynamic similarity may be anticipated.

(b) Although the authors regard the field of thermodynamics as much better adapted to consideration of the flow of compressible fluids, such as the gases, tabular and graphical facilities relating to this are provided and may be adequate for introductory purposes.

PART I

General Considerations

Chapter 1. Pertinent Properties of Fluids

1-1. Foreword: Engineering Fluid Mechanics may be said to have originated centuries ago when man began to convey needed potable water to more populous communities, from more or less distant sources of supply and via man-made channels or conduits. Subsequently he endeavored to collect and analyze relevant data whereby desired results might be accomplished with more confidence and effectiveness.

The start of the Industrial Revolution in the later part of the 18th century brought demands for power in quantities exceeding that available through the muscular efforts of man or beast. Through a rapidly developing technical competence there were evolved first the simple water wheel and later the hydraulic turbine, both for converting the energy stored in elevated water supplies into desired mechanical power. Major developments during the 19th Century included the marine propeller for the propulsion of water-borne vessels, and the centrifugal pump as a compact device for utilizing mechanical power for continuous delivery of liquids against a superior pressure. More recent developments include the fluid coupling and torque converter, fluid control and "servo" mechanisms.

As was generally the case in these earlier developmental periods, progress was largely the result of the inventive ingenuity of practical experimenters. This was followed by efforts on the part of the more analytically minded to organize and interpret experimental findings, and thereby to improve performances. Quite aptly the resulting literature was designated as that of Hydraulics. The group initially responsible for the developments during these periods was that now known as the civil engineer, but above devices have become the definite concern also of the mecanical and marine engineer.

A significant feature of the phenomena which were of concern in these earlier periods was that, with the exception of the passage of the fluid about a vessel and its propeller blades, the situations had been ones in which fluid streams passed through confining channels. A consequence was that desired results could still be secured in spite of limited information concerning the paths followed by individual particles of the fluid stream in its passage through the flow channels; i.e., the flow pattern. Also simpler mathematical facilities were thus adequate for analytical purposes.

But with the advent of the airplane in the early part of the present century, with its flight ability dependent solely on the proper amounts and distribution of the forces established on its wing and propeller surfaces by virtue of relative motion between them and a surrounding fluid (atmospheric) environment, it became quite necessary to develop analytical methods for anticipating with some degree of confidence the flow pattern assumed by the enveloping air stream. The services of the mathematical physicist, with his greater mathematical capabilities, were enlisted for devising such methods. With increase in his familiarity with the problems involved he developed a literature relating to the techniques for the analysis of these essentially kinematic aspects of flow phenomena. Although initially it pertained to the flow about the air foil (of the plane or its propeller), its significance and utility in relation to situations and devices of more general engineering concern is unquestionable.

The matters indicated above are illustrative of those involved in the studies here referred to as Engineering Fluid Mechanics; or frequently simply as Fluid Mechanics,* and to which the following material relates. But, as this is designed for the most part to meet the needs of the undergraduate engineering student, and as adequate coverage of the kinematic aspects of fluid flow requires mathematical capabilities beyond his normal range, in these aspects the material need also be regarded as introductory in nature.

It may be of present interest to the reader to note the sequence employed in the following presentation of the subject, and to have a preliminary notion of its character and purposes. The general classifications under which the material is organized are as follows -

I. General Considerations; relating to various primary concepts, measures and requirements which become of continuing concern in subsequent studies.

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* - There was for a time a trend to interpretation of this briefer title as relating essentially to the kinematic aspects of flow phenomena, but later usage has made it more inclusive.

II. Fluid Statics; wherein attention is given to those frequent situations in which the effect of the earth's gravitational force on a quiescent body of fluid is of primary concern. Related considerations are the buoyancy and stability of vessels at the surface of a liquid or submerged in any fluid environ.

III. Stream Energetics; furnishing the concepts and methods which are essential when accounting the energy distribution and transformations occurring in simpler streams and in ones to or from which energy is in transition as work. Attention is also given to an extremely useful techniques for organizing experimental data to maximum practical effectiveness.

IV. Fluid Dynamics; in which attention is given more specifically to the forces which operate to effect fluid motions and energy transformations in fluid streams. These provide further background whereby consideration may be given to operational characteristics of engineering devices such as the turbine, pump, hydraulic coupling, or propeller and ship. Engineering approximations are sufficiently recognized, but are utilized without apology, due to their virtual necessity in consequence of the extremely complex nature of the flow through such equipment.

V. Fluid Kinematics; introducing the primary concepts and basic techniques involved in description of the simpler flow patterns which may be assumed by a stream, and the method of synthesis by which more complex patterns may be anticipated for an ideal or non-viscous fluid. Here the initial material requires only the simpler mathematical facilities. However, for the interest of the reader possessing more powerful mathematical tools, their use in the conformal transformation of a composite type of flow pattern is further illustrated.

1-2. Characteristics of Fluids: From the viewpoint of fluid mechanics the primary feature identifying a fluid and distinguishing it from a rigid body, and elastic one, or even a plastic material, is its ability to conform readily to the shape of the vessel in which it is contained. But included in the general category of fluids are materials in (a) liquid and (b) gas phases. A general characteristic of a liquid is that the volume it occupies depends primarily on its quantity, and is influenced only moderately by its temperature and less so by any imposed pressure. But gas will diffuse throughout a container, due to a much smaller or even negligible attractive force between the molecules of a fluid when in the gas phase, as compared with that when in the liquid phase. In this phase the volume per unit quantity is materially influenced by both temperature and pressure. In the last feature they are describable as compressible or, if preferred, as expansible.

In these studies emphasis is primarily on the mechanics of fluids when in liquid phase. Adequate attention to phenomena with compressible fluids would require a too great excursion into thermodynamic studies, but facilities are included which, although more of handbook than analytical nature, may be sufficient for present purposes.

1-3. Basic Concepts and Units: To facilitate subsequent studies it is well first to review and perhaps clarify the fundamental concepts of time, space or size, these being truly basic in the feature of originating in our physical senses. These we customarily represent by general symbols such as T, L, and F, respectively. For quantitative evaluation of them we require means and units.

A fortunate feature when proposing to establish a unit of time is that nature provides a convenient primary unit, the year, this as established by the interval between successive attainments of the same orientation of the earth's axis of rotation with relation to our sun. Derived but more convenient subdivions of this are the (mean) day, hour, minute and second.

The earth also provides natural units of space; as for example, the distance between pole and equator. However, the immensity of this distance precludes its use as a practical unit. A fortunate feature of the distance concept is also the ease with which man could adopt and preserve a more convenient unit by a simple procedure such as scribing marks having an arbitrarily selected separation along an effectively imperishable bar. This he did when the standard meter length was selected and a platinum-iridium bar establishing this length as an international prototype was constructed with extreme care and provisions made for its storage and preservation. Other unit lengths, originating in national habits but now defined in terms of the meter length, are the mile (statute and nautical), the foot (= 0.3048 meter) and the inch.

Secondary concepts and measures deriving directly from those of space and time are those of speed or linear velocity (u), and the time-rate of change of velocity, or acceleration (a).

Although the concept of force is directly evident to man's senses, in connection with required muscular effort, means are not immediately evident whereby a particular force might be selected, packaged and stored as a standard or prototype unit force. Undoubtedly recognizing this predicament, in his 2nd Law of Motion Sir Isaac Newton ingeniously associated the concepts of force and of quantity of material in terms of the acceleration produced by the action of a force on a body and of its inertial resistance to the resultant change in its state of motion. This attribute of the body he regarded as a measure of its <u>mass</u> (m), and in this respect may be regarded also as a measure of the quantity of material comprising the body.*

One form of the 2nd Law of Motion states that, for producing a change in the state of motion of a given mass, the applied force and the resulting acceleration of the mass are directly proportional; or symbolically,

$$F = k\,m\,a\,, \tag{1-1}$$

Here <u>k</u> is simply a numerical proportionality constant, depending for proper magnitude on the force, mass, space and time units employed, but providing suitable numerical correlation when one or another combination of units is used.

As it is quite practicable to select, adopt and preserve a standard or prototype mass, and designate it as a <u>unit</u> mass, this relation will evidently serve further to define and to permit indirect evaluation of a standard unit force. The international prototype mass is a platinum-iridium cylinder which is preserved in France and is described as the <u>kilogram</u> mass. It will be recalled that the corresponding basic unit of force, designated as the <u>dyne</u> force, is one which would give a gram mass an acceleration of one centimeter per second per second (cm/sec^2), or a kilogram mass an acceleration of $1/10^5$ meters per second per second. That is, a force of 10^5 dynes, described also as a force of one <u>newton</u>, would be required to accelerate a kilogram mass at one meter per second per second. Thus, referring to

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* - This index of quantity is sufficient in analyses relating to essentially mechanical effects, but other limitations become evident when observing that equal masses of different materials are capable of producing vastly different non-mechanical effects.

eq. 1-1,

$$\text{Force, dynes} = 1.0 \times \text{mass, (gm)} \times \text{accel (cm/sec}^2)$$

$$\text{or} = 10^5 \times \text{mass (kg)} \times \text{accel (m/sec}^2)$$

$$\text{also force, newtons} = 1.0 \times \text{mass (kg)} \times \text{accel (m/sec}^2)$$

Unit combinations such as the above for which the proportionality constant (k) is unity, are said to be consistent.

In English-speaking countries the more common engineering unit of mass is the pound (avoirdupois), defined as 0.4536 kilogram. It is understandable that, when proceeding to adopt a related unit of force, it was thought that an automatically preservable unit might be secured by utilizing the earth's gravitational attraction (at a selected location) on this standard pound mass. The selected location was at 45° N. latitude and mean sea level, at which position the acceleration produced by the gravitational force is 32.174 ft/sec^2. The unit of force as so established would thus be one definable as that which would produce this acceleration when acting on a pound mass. Unfortunately (due to the very frequent resulting confusions) it became known by the same name as that of the mass unit; i.e., as the pound force. But, again referring to equation 1-1 and distinguishing the pound-force and the pound-mass by the symbols lbf and lbm,

$$\text{Force (lbf)} = 1/32.174 \times \text{mass (lbm)} \times \text{accel (ft/sec}^2),$$

where 1/32.174 is the numerical proportionality required when employing this combination of units.

Due to the convenience of consistent unitational combinations, the poundal (lbl), defined as 1/32.174 lbf, has more recently come into use as a force unit; also the slug, defined as 32.174 lbm, as a mass unit. In these units,

$$\text{Force, lbl} = 1.0 \times \text{mass, lbm,} \times \text{accel, ft/sec}^2$$

$$\text{or force, lbf} = 1.0 \times \text{mass, slugs,} \times \text{accel, ft/sec}^2.$$

The above will have indicated that the engineer has permitted himself to be burdened with an abominable multiplicity of units, due however in part to his relations both with the layman and with technology and in various lands and technical fields. An author is correspondingly perplexed when selecting a unitational system which he may best employ. In the earlier portion of this material we shall frequently employ the non-consistent pound-mass and pound-force combination, but consistent units in later portions, with the thought that the student-reader may thereby become adept in using either.

For subsequent convenience the proportionality constants required for obtaining correct numerical results when using several of the more common combinations of force, mass, space and time units are provided in Table 1-1.

Table 1-1

Unit of Force (F)	Unit of Mass (M)	Unit of Distance (L)	Unit of Time (T)	Proportionality constant, k
		CONSISTENT COMBINATIONS		
dyne	gram	centimeter, cm	second	unity
newton	kgm	meter	second	unity
poundal, lbl	pound, lbm	foot	second	unity
pound, lbf	slug	foot	second	unity
		NON-CONSISTENT COMBINATIONS		
lbf	lbm	foot	second	1/32.174
dyne	kgm	meter	second	10^5
kgf	kgm	cm	second	1/980.7

1-4. Fluid and Stream Properties: In technical terminology a property of a fluid denotes simply any pertinent attribute which is involved when describing sufficiently its physical state, or is conversely an invariable feature of the fluid when in that state.

The two properties which are rather more in the nature of causes of state are the pressure (p) and temperature (t). Others which are more consequential than casual properties but become of definite concern in fluid mechanics analyses include the density (ρ) of a fluid (or its reciprocal, specific volume, v), its viscosity (μ) and, for a liquid, its surface tension (τ). It is also essential at least to be aware of its store of internal energy (e).

For organized fluid streams significant attributes are their velocity (u) and flow density (G). Following articles relate individually to the above items.

1-5. Pressure: One probably perceives the concept of pressure more readily as the force acting per unit area (dimensionally, FL^{-2}) on a fluid surface.* Consequent units are ones such as dynes per sq. cm or newtons per sq. meter, and pounds (lbf) or poundals (lbl) per sq. ft. or per sq. inch; also in atmospheres (i.e. a representative mean or standard atmospheric pressure of 14.70 lbf/sq in or 2117 lbf/sq.ft.) Other indirect measures of pressure are indicated later.

One thinks usually of pressure as the force exerted on a unit fluid surface. But it is well to recognize it also as the result of the incessant bombardment of a confining surface by those molecules of the fluid having components of motion normal to the surface. The actual or total pressure so acting is known as the absolute pressure. However in many instances it may be sufficient to consider only the excess or the deficit of this pressure relative to atmospheric, known respectively as the gauge pressure and the vacuum.

1-6. Temperature: The concept of temperature is evident to our senses in our awareness of hotness or coldness, but in many instance may better be recognized as an index of the activity of the molecules comprising a material. The reader is of course familiar with the temperature scales of the common thermometer, with the centigrade scale having 100 divisions or degrees between a zero point at the freezing temperature of water and its boiling temperature, both for water under standard atmospheric pressure. English speaking peoples use also the Fahrenheit scale, with 32 degrees marking the freezing and 212 degrees the boiling temperatures.

Frequently it becomes necessary, however, to express temperatures relative to an absolute zero, which is known to be closely at 273°C or 460°F below the zero points of the above arbitrary scales; that is, degrees centigrade absolute, or degrees Kelvin (°K), = °C + 273, and degrees Fahrenheit absolute, or degrees Rankine (°R), = °F + 460.

1-7. Density, and Specific Volume: The volume occupied by unit mass of a fluid is known as its specific volume (v); and its reciprocal, or the mass per unit volume, as its density (ρ ,rho). Habitually the density is quoted for liquids, and the

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* - The word has been used also to denote the total force on any extended surface. The above significance is however the more generally accepted one.

specific volume for gases. Customary density units are the composite ones of gm/ cu.cm, lbm/cu.ft or slugs/cu.ft, and their reciprocals for specific volume.

The density of a liquid is appreciably influenced by its temperature but rather negligibly by a considerable change of pressure. For gases under moderate pressures it is materially affected by both, in accordance with the relations

$$v \text{ (cu. ft/lbm)} = \frac{1545 \times T\ (^{o}R)}{M \times p\ (\text{lbf/sq. ft})}$$

$$\text{or} \quad v \text{ (cu. ft/slug)} = \frac{48.020 \times T\ (^{o}R)}{M \times p\ (\text{lbf/sq. ft})}$$

$$\text{and} \quad v \text{ (cu. m/kgm)} = \frac{8311 \times T\ (^{o}K)}{M \times p\ (\text{ne/sq. m})} \tag{1-2}$$

where M = formula mass (or "molecular weight") of gas; as, for example, 28.97 for dry air. These relations become definitely inexact at higher pressures and temperatures.

The numerical ratio between the density of any fluid and that of water if the fluid is liquid at 39°F (4°C, or that of air at 68°F (20°C) and 14.70 lbf/sq. in. if a gas, is known as its specific gravity and is a quite common indirect index of density. At these states the density of water is 62.42 lbm or 1.940 slugs/cu. ft. and 1 gm/cu. m. and that of air is 0.07519 lbm/cu.ft (1.205 kg/cu.m).

In various industries other numerical indices are used for indirect evaluation of the density of liquids; notably in the American petroleum industry, in which

$$\text{deg. A.P.I} = \frac{141.5}{\text{sp. gr, } 60/60^{o}F} - 131.5,$$

where sp.gr. 60/ 60°F denotes the ratio of the specific gravity of the oil at 60°F to that of water at that temperature (0.9988, or effectively unity).

In some instances it is required to consider the local gravitational force on, or weight of, unit volume of a fluid, or its specific weight (γ). Recall that, paralleling eq. 1-1,

$$\text{F grav.} = k \times \text{mass} \times \text{accel. grav.}$$

$$\text{or} \quad \text{weight} = k \times \text{mass} \times g, \tag{1-3}$$

where g denotes the acceleration produced by the local gravitational force and k has values such as 1.0, 10^5, 1/32.174 et cetera as various force, mass and space units are employed, as listed in Table 1-1. Although the weight of a material of given mass varies rather negligibly with its location on or even considerable above the earth's surface, its marked decrease on passage into space requires recognition in these times when the school boy talks glibly of space ships and the scientist gives them serious consideration.*

1-8. Viscosity. To a degree analogous to the resistance to shearing action which is exhibited by elastic or plastic materials, fluids exhibit also a resistance to relative motions within lamina or between adjacent lamina in a fluid stream. The index employed in evaluating this characteristic is known as the absolute, or dynamic viscosity of the fluid, and is represented by the symbol μ (mu).

One readily recognizes this effect if endeavoring to move even a thin plate through a "sticky" liquid, and may properly attribute it to cohesions between contiguous molecules. But even for liquids, and more positively so for gases, it needs be recognized as attributable also to the unceasing activity of the molecules comprising the fluid and their associated passages between and into adjacent planes. In this respect the action is not unlike the effect, in modifying their rates of advance, that would be produced b y an active barrage of projectiles exchanged between and becoming embedded in the walls of passing vehicles.

The manner of evaluating the influence of viscosity is indicated in Figure 1-1, representing a lamina between two planes in any fluid stream. The planes are of equal area (A) and separated by distance dL. If the lower plane were moving at velocity u, but the upper at u du, equal but opposing forces F would

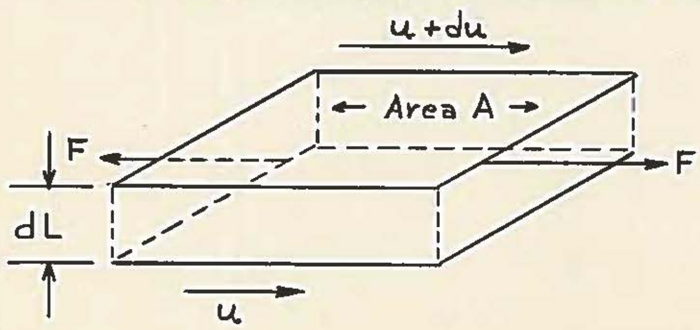

Fig. 1-1

* - In this connection note that g may very conveniently be given the dimensional significance of local weight per unit mass; also that mass may be expressed as the ratio between local weight and local gravitational acceleration. Although correspondingly the symbol wt/g has frequently been employed in the literature for denoting mass, the authors prefer its direct notation. But note that (wt, lbf)/(g, ft/sec^2) does in fact express mass in slugs.

be required to maintain the relative motion. For such laminar flow the requisite force is found to be directly proportional to the area and the velocity difference but inversely so to the distance between the planes, and also is taken to be proportional to the property described above as the viscosity of the fluid. That is,

$$F \propto \mu A(du/dL), \text{ or } \quad \mu = C\,\frac{F/A}{du/dL} \quad \text{(By definition)}, \qquad (1\text{-}4)$$

where du/dL expresses the rate of deformation of the fluid lamina, and C is such a numerical proportionality constant as will suitably accomodate the combination of units employed.*

Note in Equation 1-4 that, in terms of the primary time, space and force or mass concepts or dimensions (T, L and F or M,) viscosity has the significance either of -

$$\frac{F\,L^{-2}}{LT^{-1}/L} \text{ and thus } FTL^{-2}, \text{ or of } \frac{(MLT^{-2})/L^2}{LT^{-1}/L} \text{ and thus } MT^{-1}\,L^{-1}$$

These dimensional formulas are the basis of composite units employed for viscosity evaluation. Such are those of dyne-seconds per sq.cm. or alternatively, grams per second-centimeter, either being known also as the viscosity in poises.

Parallel English units are those of (a) Poundals-seconds per sq. ft. or pounds lbm per second-foot, and (b) pounds (lbf)-seconds per sq. ft. or slugs per second-foot. Figure 1-2 indicates for a number of more common fluids the manner of variation of their dynamic viscosity with temperature, in the above units and all when at moderate pressure.

In lubrication studies a still further unit has been used, describable as the viscosity in pounds (lbf)-seconds per sq. in. and known also as the viscosity in reyns.

In analyses relating to the flow of fluids the quotient of the dynamic viscosity divided by the density (μ/ρ) appears so regularly that it is conveniently regarded as a significant supplementary property. Recalling that density has the primary dimension ML^{-3}, this ratio has thus the dimensional formula $(ML^{-1}T^{-1})/(ML^{-3})$ or L^2T^{-1}.

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* - A fluid for which the viscosity is constant for all deformation rates would be said to be Newtonian. Most fluids exhibit some variation, but not sufficient to require engineering attention when flow is laminar.

The property is distinguished as the <u>kinematic</u> viscosity (ν , nu) of the fluid, as the force concept does not appear in its formula. Again the units for its evaluation are composive ones deriving from its formula, with corresponding evaluations <u>in sq. cm per second</u> or in <u>sq. ft. per second.</u> Figure 1-3 indicates for a number of fluids the manner of variation of ν with temperature, again at moderate pressures.

Instruments are available for direct determination of the dynamic viscosity of fluids(see art.9-6) but for liquids indirect means are customarily employed for determination of the kinematic. These take the form of containers, of specified dimensions, which may be drained through a vertical tube, of definite diameter and length. As the time required for the efflux of a specified quantity of the liquid is a direct function of its kinematic viscosity, it is expressed in terms of that time. An accepted instrument of this type is the Saybolt viscometer, and data determined thereby are described as the kinematic viscosity in <u>seconds Saybolt.</u> Fig. 1-4 provides conversion data from several such indirect measures to the basic units of sq. cm or sq. ft per second.

1-9. <u>Surface Tension:</u> The characteristic known as the <u>surface tension,</u> τ , of a liquid is a measure of the force of mutual attraction between the molecules that comprise the liquid, with particular reference however to the effect of the free surface. The tension is evaluated in the terms of force per unit length. Thus, if a line of unit length is imagined in the surface of a liquid, the surface on either side of the line exerts a force on the opposite side equal in magnitude to the surface tension. Tau becomes of fluid mechanic concern when it operates to influence the form of liquid surfaces, as later noted in article 2-5.

1-10. <u>Unitary and Dimensional Homogeneity:</u> In his varied responsibilities it is necessary that the engineer be competent not only to analyze the situations he encounters and to formulate his conclusions, but also to correctly translate such conclusions into numerical results. In the latter connection it is not too surprising that with the varied and frequently inconsistent units (art. 1-3) in which his data may be expressed or results may be desired, the English-speaking engineer may easily dissipate an undue amount of time in unitary confusions.

A simple aid in minimizing these lies in the realization that, for insuring unitary and numerical validity, the same units and the same powers of those units must in effect appear on <u>both</u> sides of any relation associating physical variables i.e., the relation must be <u>unitarily homogeneous.</u> To illustrate in elementary manner, presume that it is desired to express, in foot-pounds (lbf), the work required to transport an object through a given distance, evaluated in miles, against a restraining force expressed also in pounds.

Writing the relation associating these items, but indicating first such units as would provide the required homogeneity

$$\text{work (ft-lbf)} = \text{force (lbf)} \times \text{distance (ft)}.$$

Introducing both the alien units and such numerical multipliers as serve to translate the alien units to homogeneous ones,

$$\text{work (ft-lbf)} = \text{force (lb)} \times \text{distance (miles)} \times \frac{\text{feet}}{\text{mile}}$$

$$\text{and thus} = \text{force (lb)} \times \text{distance (miles)} \times 5280$$

Here the indicated unit-ratio and associated number, or conversion factor is simply one by which the evaluation of a physical magnitude in one (alien) unit is converted to that in a preferred unit.

The conversion tables of the following article provide the requisite multiplying factors whereby, for such units and unit combinations as will be encountered in following material, evaluations expressed in one unit or combination may be translated to that in a preferred one.

In some respects as a corollary to the above necessity for unitary homogeneity but in other respects as an independent and more fundamental requirement, in any valid physical relation on the primary <u>concepts</u> of space (L), time (T), force (F) or mass (M) must also appear to the same powers on both sides of the relation when the variables which are involved have been expressed in terms of their dimensional formulas. That is, for even the <u>possibility</u> of propriety, any equation purporting to express the relationship between the variables influencing a physical phenomenon must be <u>dimensionally homogeneous.</u>

The accompanying Table 1-2 collects the major quantities of fluid-mechanic concern, their general character and fundamental dimensions.

Table 1-2
Character and Fundamental Dimensions
of Fluid Mechanics Quantities

Quantity	General Character	Fundamental Dimensions in terms of M, L & T	F, L & T
Area	product of two linear dimensions	L^2	
Volume	product of three linear dimensions	L^3	
Velocity, linear	distance per unit time	$L T^{-1}$	
Velocity, angular	linear vel. per unit radius.	T^{-1}	
Acceleration	(change of) velocity per unit time, or force per unit mass	$L T^{-2}$, or FM^{-1}	
Momentum	mass times velocity	$M L T^{-1}$	FT
Density	mass per unit volume	$M L^{-3}$	$FL^{-4}T^2$
Force (& weight)	mass times acceleration	$M L T^{-2}$	F
Pressure	force per unit area	$M L^{-1}T^{-2}$	FL^{-2}
Specific weight (w)	weight per unit volume	$M L^{-2}T^{-2}$	FL^{-3}
Energy	force times distance	$M L^2 T^{-2}$	FL
Energy per unit mass		$L^2 T^{-2}$	
Power	energy per unit time	$M L^2 T^{-3}$	FLT^{-1}
Torque	force times moment arm	$M L^2 T^{-2}$	FL
abs. viscosity	$\frac{\text{force per unit area}}{\text{velocity per unit separation}}$	$M L^{-1}T^{-1}$	$FL^{-2}T$
Kinematic viscosity	abs. visc./density	$L^2 T^{-1}$	
Surface tension	force per unit length	$M T^{-2}$	FL^{-1}

To illustrate let us verify the dimensional homogeneity of an equation which (although predictable analytically) will be presumed to have been developed from experimental investigations. The investigations are ones for determining the influences on the pressure decrease $(-\Delta p)$, in a laminar type of flow of a fluid stream, which are exerted by variation of the duct length (l) and transverse area (A) and by the density, viscosity and mass-rate of flow ($\dot{m}$) of the fluid. The experimental evidence indicates that these would be suitably related by an equation of the form -

$$-\Delta p \propto \dot{m} l \mu / \rho A^2$$

or $$-\Delta p = C \dot{m} l \mu / \rho A^2$$

Here C is that numerical constant which serves to reconcile possibly inconsistent units used in evaluation of the variables and also to account for influences of the character of the duct.

Recalling the dimensional formulas for the several variables, and introducing and combining them,

$$FL^{-2}, \text{ or } ML^{-1}T^{-2}, \longleftrightarrow \frac{(MT^{-1})\,(L)\,(FTL^{-2} \text{ or } ML^{-1}T^{-1})}{(ML^{-3})\,(L^4)}$$

$$\text{and } FL^{-2} \longleftrightarrow (FL^{-2}), \text{ or } ML^{-1}T^{-2}$$

The equation is thus seen to be dimensionally homogeneous, and to that extent valid for associating the relevant variables.

A useful supplemental feature of a unitary or dimensional survey of a proposed empirical relation is that, if found to be non-homogeneous, this may provide a valid clue to the unsuitable inclusion of an irrelevant variable or the omission of a relevant one.

Although at this point it will not be at all self-evident, it will later be seen that an astonishing facility enabling the utilization of limited experimental data to quite general purposes may be provided, in part, by the collection of the relevant variables into groups or combinations which are <u>non-dimensional</u>, or <u>"dimensionless"</u>; that is, ones having the dimensional formulas $F^0T^0L^0$ or $M^0T^0L^0$. In the relation analyzed above such groupings would be $\dfrac{\Delta p \, \rho \, A^{5/2}}{L \, \dot{m}^2}$ and $A^{1/2}/\dot{m}$.

An inherent and convenient feature of such a non-dimensional group is that its <u>numerical</u> magnitude will be unaffected by the time, space, force and mass units employed in evaluating its variables, <u>so long as the system of units employed is consistent</u>; that is, one for which the proportionality constant <u>k</u> (table 1-1) is unity.

1-11. <u>Conversion Tables:</u> The following tables provide the magnitudes of such numerical multipliers as will be required for unitary conversions in following analyses or problems. As is indicated, the arrangements are such that the number in any space is that by which an item expressed in the unit of its column shall be multiplied for evaluation of the item in the unit of its line.* For convenience the dimensional formulas for the item of each table are also quoted.

* - More extended tables appear in the engineering handbooks, but in some in a reciprocal type of arrangement. Note in these connections that, for example, as (or although) 1 mile = 5280 feet, the number evaluating a distance in miles is 1/5280 th of that evaluating it in feet.

Table I-3. Unitary Conversion Multipliers

Length (L)

to obtain number of ↘ Multiply number of by	feet	miles, stat.	meters
feet	1	5,280	3,281
miles (statute)	$\frac{1.894}{10^4}$	1	$\frac{6.214}{10^4}$
meters	.3048	1,609	1

Area (L^2)

	sq. ft	square miles	square meters
sq. feet	1	2.788 x 10^7	10.76
sq. miles	$\frac{3.587}{10^8}$	1	$\frac{3.861}{10^7}$
sq.meters	$\frac{9.290}{10^2}$	2.590 x 10^6	1

Volume (L^3)

to obtain number of ↘ Multiply number of by	cu.ft	US gal.	liters
cubic feet	1	.1337	$\frac{3.531}{10^2}$
U.S. gallons	7.481	1	.2642
liters (.001 cu.m)	28.32	3.785	1

Volume rate (L^3T^{-1})

	$\frac{\text{cu.ft}}{\text{sec}}$	$\frac{\text{gal}}{\text{min}}$	$\frac{\text{liters}}{\text{min}}$
cu.ft/sec.	1	$\frac{2.228}{10^3}$	$\frac{3.886}{10^4}$
US gal/min.	448.9	1	.2642
liters/min.	1,699	3.785	1

Linear velocity (LT^{-1})

to obtain number of ↘ Multiply number of by	$\frac{\text{ft}}{\text{sec}}$	$\frac{\text{miles}}{\text{hr}}$	knots
feet/second	1	1.467	1.689
miles/hour	.6818	1	1.152
knots (6080 $\frac{\text{ft}}{\text{sec}}$)	.5921	.8684	1

Angular velocity (T^{-1})

	rpm	$\frac{\text{radians}}{\text{sec}}$	$\frac{\text{deg.}}{\text{sec}}$
rev./min	1	9.549	.1667
radians/sec	.1047	1	$\frac{1.745}{10^2}$
degrees/sec	6.000	57.30	1

Mass (M, or FT^2L^{-1})

to obtain number of ↘ Multiply number of by	lbm	slugs	kg
pounds (lbm)	1	32.17	2.205
slugs (g-pounds)	$\frac{3.108}{10^2}$	1	$\frac{6.852}{10^2}$
kilograms	.4536	14.59	1

Density (ML^{-3}, or FT^2L^{-4})

	$\frac{\text{lbm}}{\text{cu.ft}}$	$\frac{\text{lbm}}{\text{gal(US)}}$	$\frac{\text{kg}}{\text{cu.m}}$
lbm/cu.ft	1	7.481	$\frac{6.243}{10^2}$
lbm/gal(US)	.1337	1	$\frac{8.345}{10^3}$
kg/cu.meter	16.02	119.8	1

Force (MLT^{-2}, or F)

to obtain number of ↘ Multiply number of by	lbf	$\frac{\text{lbm, ft}}{\text{sec}}$	newtons
pounds (lbf)	1	$\frac{3.108}{10^2}$	.2248
poundals (lbl); (lbm.ft)/sec	32.17	1	7.233
newtons (10^5 dynes)	4.448	.1383	1

Torque (ML^2T^{-2}, or FL)

	lbf. ft	$\frac{\text{lbm}}{\text{sec}}$ x rpm x ft^2	newton meter
lbf.ft	1	$\frac{3.254}{10^3}$	.7376
$\frac{\text{lbm}}{\text{sec}}$ x rpm x ft^2	307.2	1	226.9
newton meters	1.356	$\frac{4.407}{10^3}$	1

Table 1-3 continued

Viscosities

Absolute, or Dynamic ($ML^{-1}T^{-1}$, or FTL^{-2})

to obtain number of \ Multiply by number of	slugs/sec.ft; lbf.sec/ft²	lbm/sec.ft; lbl.sec/ft²	gm/sec.cm; dyne sec/cm²
slugs/sec.ft or lbf.sec/ft²	1	$\frac{3.108}{10^2}$	$\frac{2.089}{10^3}$
lbm/sec,ft or lbl.sec/ft²	32.17	1	$\frac{6.720}{10^2}$
poises; gm/sec.cm or dyne sec/cm²	[illegible]	14.88	1

Kinematic (L^2T^{-1})

	sq.ft/sec	stokes; sq.cm/sec	poises/lbm/ft³
sq.ft/sec	1	$\frac{1.076}{10^3}$	$\frac{6.719}{10^2}$
stokes; sq.cm/sec	929.0	1	62.43
poises/lbm/ft³	14.88	$\frac{1.602}{10^2}$	1

Pressure ($ML^{-1}T^{-2}$ or FL^{-2})

to obtain number of \ Multiply by number of	lbf/ft²	lbf/in²	newtons/sq. m	atmos., (std)	feet of water *	inches of Hg. *
lbf per sq. foot	1	144	$\frac{2.089}{10^2}$	2,117	62.43	70.73
lbf per sq. inch	$\frac{6.944}{10^3}$	1	$\frac{1.450}{10^4}$	14.70	.4335	.4912
newtons/sq. meter; (dyne/cm²)/10	47.88	6.895×10^3	1	1.013×10^5	2.989×10^3	3.386×10^3
atmospheres, std.	$\frac{4.725}{10^4}$	$\frac{6.804}{10^2}$	$\frac{9.869}{10^6}$	1	$\frac{2.950}{10^2}$	$\frac{3.342}{10^2}$
feet of water; at 39° F. *	$\frac{1.602}{10^2}$	2.307	$\frac{3.346}{10^4}$	33.90	1	1.133
inches, mercury; at 32° F. *	$\frac{1.414}{10^2}$	2.036	$\frac{2.953}{10^4}$	29.92	.8826	1

*. The multipliers indicated in connection with these items are in principle correct only if the liquid is in a gravitational field having the "standard" strength of 1 lbf per lbm, as the items express actually the ratio $\Delta p/\gamma$ (where γ = local specific weight of liquid; eq. 2-1a) and are correspondingly incompatible in dimensions with those of pressure.

Energy (ML^2T^{-2}, or FL)

to obtain number of \ Multiply by number of	ft.lbf	hp-hrs	Btu	newton meters	kw-hr	cal.
foot pounds, or (lbf/ft²) x cu.ft	1	1.980×10^6	778.2	.7376	2.656×10^6	3.088
horsepower hours	$\frac{5.051}{10^7}$	1	$\frac{3.930}{10^4}$	$\frac{3.725}{10^7}$	1.341	$\frac{1.560}{10^6}$
Btu	$\frac{1.285}{10^3}$	2,544	1	$\frac{9.475}{10^4}$	3,412	$\frac{3.968}{10^3}$
newton meters, or joules (closely)	1.356	2.684×10^6	1,055	1	3.601×10^6	4.187
kilowatt hours	$\frac{3.766}{10^7}$	.7457	$\frac{2.930}{10^4}$	$\frac{2.777}{10^7}$	1	$\frac{1.163}{10^6}$
calories	.3240	6.413×10^5	252.0	.2388	8.600×10^5	1

Table 1-3 continued

Energy per unit mass (L^2T^{-2}, or FLM^{-1})

to obtain number of \ Multiply number of \ by	$\frac{ft.lbf}{lbm}$; $\frac{lbf/ft^2}{lbm/ft^3}$	$\frac{ft.lbf}{slug}$	$(\frac{ft}{sec})^2$	$(\frac{miles}{hour})^2$	$\frac{Btu}{lbm}$	$\frac{cal}{gm}$	newton-meters per gm
ft.lbf/lbm, and (lbf/ft^2)(ft^3/lbm)	1	$\frac{3.108}{10^2}$	$\frac{1.554}{10^2}$	$\frac{3.341}{10^2}$	778.2	1,401	.3347
ft.lbf/slug	32.17	1	0.500	1.075	2.504 x 10^4	4.507 x 10^4	10.77
(feet/second)^2	64.35	2.000	1	2.150	5.008 x 10^4	9.014 x 10^4	21.54
(miles/hour)^2	29.92	.9302	.4651	1	2.329 x 10^4	4.192 x 10^4	10.01
Btu/lbm	$\frac{1.285}{10^3}$	$\frac{3.994}{10^5}$	$\frac{1.997}{10^5}$	$\frac{4.296}{10^5}$	1	1.800	$\frac{4.300}{10^4}$
cal/gm	$\frac{7.138}{10^4}$	$\frac{2.218}{10^5}$	$\frac{1.105}{10^5}$	$\frac{2.387}{10^5}$	.5556	1	$\frac{2.388}{10^4}$
newton meters/kg	2.989	$\frac{9.289}{10^2}$	$\frac{4.643}{10^2}$	$\frac{2.988}{10^2}$	2,326	4,187	1

Power (ML^2T^{-3}, or FLT^{-1})

to obtain number of \ Multiply number of \ by	$\frac{ft.lbf}{sec.}$	horse-power	$\frac{gal(US)}{min.}$ x $\frac{lbf}{in^2}$	lbf.ft x rpm	$\frac{Btu}{min.}$	kw
ft.lbf/second	1	550	.3208	.1047	12.97	737.5
horsepower	$\frac{1.818}{10^3}$	1	$\frac{5.931}{10^4}$	$\frac{1.904}{10^4}$	$\frac{2.358}{10^2}$	1.341
$\frac{gal(US)}{min}$ x $\frac{lbf}{in^2}$	3.117	1.715 x 10^3	1	.3264	40.43	2,299
lbf.ft x rpm	9.549	5.252 x 10^3	3.063	1	123.9	7,042
Btu/min	$\frac{7.712}{10^2}$	42.40	$\frac{2.474}{10^2}$	$\frac{8.074}{10^3}$	1	56.89
kilowatts	$\frac{1.356}{10^3}$	.7455	$\frac{4.351}{10^4}$	$\frac{1.420}{10^4}$	$\frac{1.758}{10^2}$	1

1-12 Problems:

I-1. For the following liquids determine the density in pounds per cu. ft. and in pounds per U. S. gallon (231 cu. in.), and also their specific weight in a gravitational field in which one pound mass weighs one pound:-

(a) fresh water at 60° and 180°F;
(b) alcohol of a specific gravity of 0.79;
(c) oil of a gravity of 20 deg. A.P.I.

I-2. The viscosity of an oil is reported to be 150 seconds Saybolt Universal at 60°F, and its gravity as 36 deg. A.P.I. Determine the kinematic and absolute viscosities, in sq. ft./sec and lb_m/sec-ft respectively. Ans. 3.43×10^{-4}, 0.0181

I-3. For the oil of problem 2 compute the force in pounds required to effect motion at a relative velocity of 10 ft/sec between surfaces of 72 sq. in. area when separated a distance of 0.5 foot, the oil being presumed to adhere to the surfaces and to move without turbulence in the space between. Ans. 0.00562

I-4. For water at the temperature at which $\gamma = 0.005\ lb^f/ft$ compute the rise of its meniscus in a clean glass tube of 1/8th inch (0.0104) Ans. 0.37

I-5. Test of a centrifugal pump at given conditions show that for furnishing mechanical effects associated with the delivery of the fluid, in the amount of 200 ft-lb per pound, work input to the pump shaft in the amount of 280 ft-lb is required. If the difference goes primarily to increase of the internal energy of the fluid, and the receipt of one Btu of energy (= 778 ft-lb) internally will warm one pound of the fluid through one deg. Fahr., what temperature rise of the fluid while enroute through the pump is consistent with the above test data?

Ans. 0.103°F

I-6. Compute the specific weight of a liquid the density of which is 56 lb/cu. ft if it is located at a point where the acceleration of gravity is 29.5 ft/sec^2.

Ans. 51.35

I-7. Justify the alternative designations of absolute viscosity in terms of (force x time)/area and of mass/(time x length), and also of the specific unit combinations of (lbl - sec)/ft^2 and 16/(sec - ft), and of (lb-sec)/ft^2 and slugs/(sec - ft).

I-8. Justify the designation of kinematic viscosity in terms of the units sq.ft./sec.

I-9. A useful relation which is subsequently developed is to the effect that -

Power required to deliver a liquid through a pipe against resisting pressure $= \dfrac{\text{(Mass delivered per unit time} \times \text{resisting pressure)}}{\text{density of liquid}}$

Verify the dimensional homogeneity of the relation, and indicate suitable units for evaluation of the items to the right in order that the result would be secured directly in ft-lb per minute.

I-10. Verify that the quotients $\dfrac{\text{force}}{\text{density} \times \text{area} \times (\text{velocity})^2}$ and $\dfrac{\text{size} \times \text{velocity} \times \text{density}}{\text{absolute viscosity}}$ are dimensionless.

I-11. Verify the multiplier 1.467 which appears in table I-3 for conversion from velocity in miles per hour to that in ft/sec.

I-12. Verify the dimensional propriety of the product of volume rate of flow x pressure as a measure of power, and the multiplier $5.831/10^4$ of table I-3 for conversion to an evaluation of power in horsepower from an initial quotation of flow rate in gal(US)/min and of pressure in lb_f/in^2.

I-13. Employing solely the conversions 1 ft = 30.48 cm, 1 lb_f = 445×10^3 dynes and 1 lb_m = 453.6 gm verify the various conversion factors for absolute viscosity which appear in table I-3. Also evolve the multiplier for conversion from a value of μ in Reyns to its value in lb_m/sec-ft.

Part II

Fluid Statics

Chapter 2. Hydrostatic Pressure

2-1. Foreword: In many situations one requires information concerning the magnitude of the pressure at a given point in a fluid mass which is virtually at rest, in a condition of equilibrium in the gravitational field, and of the manner of variation of the pressure with depth in the fluid mass. Typical cases include the pressure and resulting forces on the surfaces of submerged vessels, the pressure existing in various locations in a liquid reservoir, and also the pressures existing at various altitudes in the atmosphere. Following articles relate to these and associated situations..

2-2. Pressure at a Point: In a body of liquid at rest the absence of motion obviates those shearing forces which motion and the fluid viscosity would otherwise introduce.

A result is that -

(A) Considering any surface (actual or imagined) in a fluid at rest, the force acting at any given point of the surface must act perpendicularly to the surface at that point.

In order that no motion shall exist in the fluid field it is further necessary that no unbalanced force exists at any point or in any direction at that point in the field. A result as it relates to the force per unit area or pressure at the point is evident on considering some minute mass of fluid about a given point. Fig. 2-1 illustrates the mass being taken for convenience to be wedge-shaped in form. In order that there be no motion of the mass,

$F_2 = F_1 \cos \alpha$,

where α is the angle between any pair of planes and also between the forces acting perpendicularly to those planes. Plane 2 might be taken as the vertical one, as indicated, or as the horizontal one*

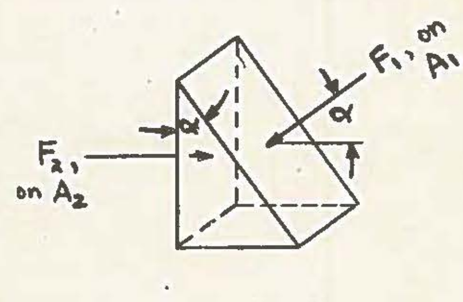

Figure 2-1

* - For a finite wedge the force at its base would exceed that on a vertical face, due to the weight of the wedge, but for the infinitesimal wedge at a point that weight becomes negligible in compassion with any finite forces which may otherwise be imposed.

In either event,

$$F_2 = F_1 \cos\alpha \text{ , and}$$

$$A_2 = A_1 \cos\alpha \text{ ,}$$

where A_2 and A_1 are the areas of the several planes. Thus

$$p_2 = F_2/A_2 = F_1 \cos\alpha \,/\, A_1 \cos\alpha = F_1/A_1 = p_1 \text{ .}$$

As α might be of any magnitude and plane 2 may be oriented in any corresponding direction, the general conclusion is that -

(B) <u>The pressure at any point in a stagnate fluid field is the same in all directions.</u>*

2-3. <u>Pressure Variation with Depth:</u> Figure 2-2 represents a lamina of fluid of density ρ lying between horizontal planes separated by vertical distance <u>dz</u>, and forming a component parcel at the base of a vertical column of fluid of transverse area <u>A</u> and extending distance $\underline{Z} - \underline{Z_0}$ to a <u>free</u> surface, if a liquid. In the latter case a supplementary pressure <u>p'</u> may also be extraneously imposed on this free surface.

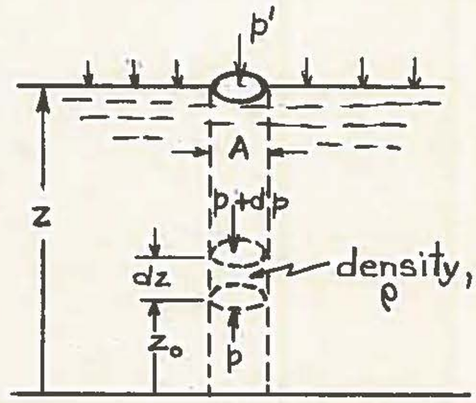

Figure 2-2

The mass of the parcel is thus ρ A dz, and its weight g ρ A dz if force and mass are expressed in consistent units. Regarding upwardly-directed forces and distances as positive in sense, for static equilibrium of the parcel the distribution of pressure and gravitational forces must be such that -

$$pA - (p + \Delta p)A - g(\rho A \ dz) = \text{zero}$$

or

$$dp = -g \rho \, dz \text{ in consistent units,} \qquad (2\text{-}1)$$

$$\text{or} = -(g/32.17) \rho \, dz \text{ if pressure is in}$$

lbf/sq. ft. and density in lbm/cu. ft.

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* - In this respect the concept of pressure is necessarily to be regarded as a <u>scalar</u> rather than a <u>vectorial</u> quantity. But as a force, evaluated by the product p x A, is directional and thus vectorial, the area concept is also suitably regarded as vectorial.

Thus, for the entire column,

$$p - p' = g\int_{z_o}^{z} \rho\, dz\ , \quad \text{and} \quad g\rho\,(z - z_o) \ \text{ if } \rho \text{ is constant.} \tag{2-1a}$$

But if the constant density was instead expressed indirectly in terms of a (local) specific weight,

$$p - p' = \gamma\,(z - z_o)$$

To state these conclusions as a third proposition but relating more particularly to a liquid of uniform density and specific weight,

(C) The pressure at any point below the free surface in a stagnate body of liquid exceeds that at the surface by the product of the depth of submergence times the specific weight of the liquid.

A closely associated observation is that, for conformity with above proposition (B), the pressure acting horizontally at any point within or in the periphery of the parcel and in its horizontal plane must also be the same. Extending this consideration to all contiguous points or to more distant ones but in the same fluid and in the same horizontal plane* -

(D) The pressure is the same throughout a given horizontal surface in a stagnate, homogeneous and interconnected body of fluid.

The four foregoing propositions provide the basis for most analyses pertaining to fluid statics.

2-4. Atmospheric Pressure; Pressure Altimetry: The depth of the earth's atmospheric envelope and its mean density are such that, at mean sea level, the weight of the over-lying column of air causes on the average the familiar "standard" atmospheric pressure of 14.70 lbf/sq. in, 2117 lbf/sq. ft or 10.13 newtons/sq. cm.

Due to the relative ease and precision with which this pressure may be measured by means of the mercurial barometer, it is recalled that it is also expressed quite commonly in indirect terms of the length of that vertical column of mercury (at 0°C) which would exhibit the same pressure difference between top and base. By Equation 2-1, and at the corresponding mercurial density of 26.39 slugs/cu. ft, this length

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* - Although not of usual fluid-mechanic concern, it may be noted that a horizontal surface is not a flat but more nearly a spherical one, of some 7920 miles diameter at sea level at the equator. More specifically, it is a surface which at all points is normal to the resultant of the mutually attractive force acting between a body and the center of mass of the earth, and of a centrifugal force which is normal to the axis of rotation of the earth due to the circular motion of the body about that axis and proportional to its distance from the axis. At the equator the latter force has the (maximum) magnitude of only about .0034 lbf/lbm.

becomes 2117/(32.174x26.39) or 2.493 ft, 29.92 inches, or 76.00 cm.

Even at sea level the actual atmospheric pressure is somewhat variable, as it is never wholly stagnate and for other reasons that are not fully known. But unless otherwise specified the above figure of 14.70 lbf/sq. in. will be employed in the following examples or problems.

Equation 2-1a serves also to ascertain the vertical distance ascended through the atmosphere if the concurrent manner of change of pressure and density is determined.

That is, $$z - z_o = (1/g) \int_p^{p_o} \rho^{-1} dp.$$

The technique is known as that of barometric or pressure altimetry, and is employed quite precisely by the meteorologist through the release of balloons carrying instruments which measure the pressure and temperature (and humidity) as ascent proceeds. Charts are available whereby the above integration may in effect be performed graphically.

A convenient but much less precise procedure is used by the aviator or mountain climber, in which only an (absolute) pressure gauge and possibly a thermometer are carried , but only a presumably representative manner of vertical variation of temperature and thus of density is assumed. More specifically, and although considerable departures frequently occur, the temperature of the atmosphere within the tropopause* is presumed to decrease or "lapse" on ascent at a constant rate of about 3.5°F per 1000 feet. The formula mass of the air therein is taken as 28.97. Thus, by Equation 1-2, $\rho = p/53.34T$.** Introducing this relation in Equation 2-1

$$dp = -(g/32.17) \times (p/53.34\,T)\,dz$$

$$\text{or}\quad dp/p = -(g/32.17)x(1/53.34)(dT/T)(dz/dT).$$

Thus, at $g_{ave} = 32.17$ and $(dT/dz) = -.0035$ deg/ft,

$$0.187\,dp/p\ (\text{or } d \ln p) = d \ln T, \text{ or } T/T_o = (p/p_o)^{.187}$$

* - The troposphere is the portion of the atmosphere extending from the earth's surface to a tropopause at some 5 miles elevation at the poles to some 7.5 miles at the equator, and in which layer active mixing exists.

** - Although this relation may appear to be dimensionally non-homogeneous, it is however homogeneous as item 53.34 has in fact the dimensional significance of ft.lbf/lbm, °R.

Correspondingly, as $T-T_o = -.0035(z-z_o)$,

$$(z - z_o), \text{ ft} \cong 286\ T_o,\ {}^oR \left[1-(p/p_o)^{.187}\right] \text{ or } 286\ T\left[(p_o/p)^{.186}-1\right]; \qquad (2\text{-}2)$$

where items with and without subscript refer respectively to lower and upper positions.

2-5. Manometric Pressure Measurements: The term manometer denotes in general an assembly of several tubes which are connected at one end, or of a tube and reservoir, and which contains at least two non-mixing fluids of different densities. Figure 2-3 represents several typical arrangements. These are such that measurement of the vertical distance (ΔZ) between the surfaces of separation of the fluids serves indirectly to determine the difference between the pressures existing in the regions to which the other ends of the tubes connect.

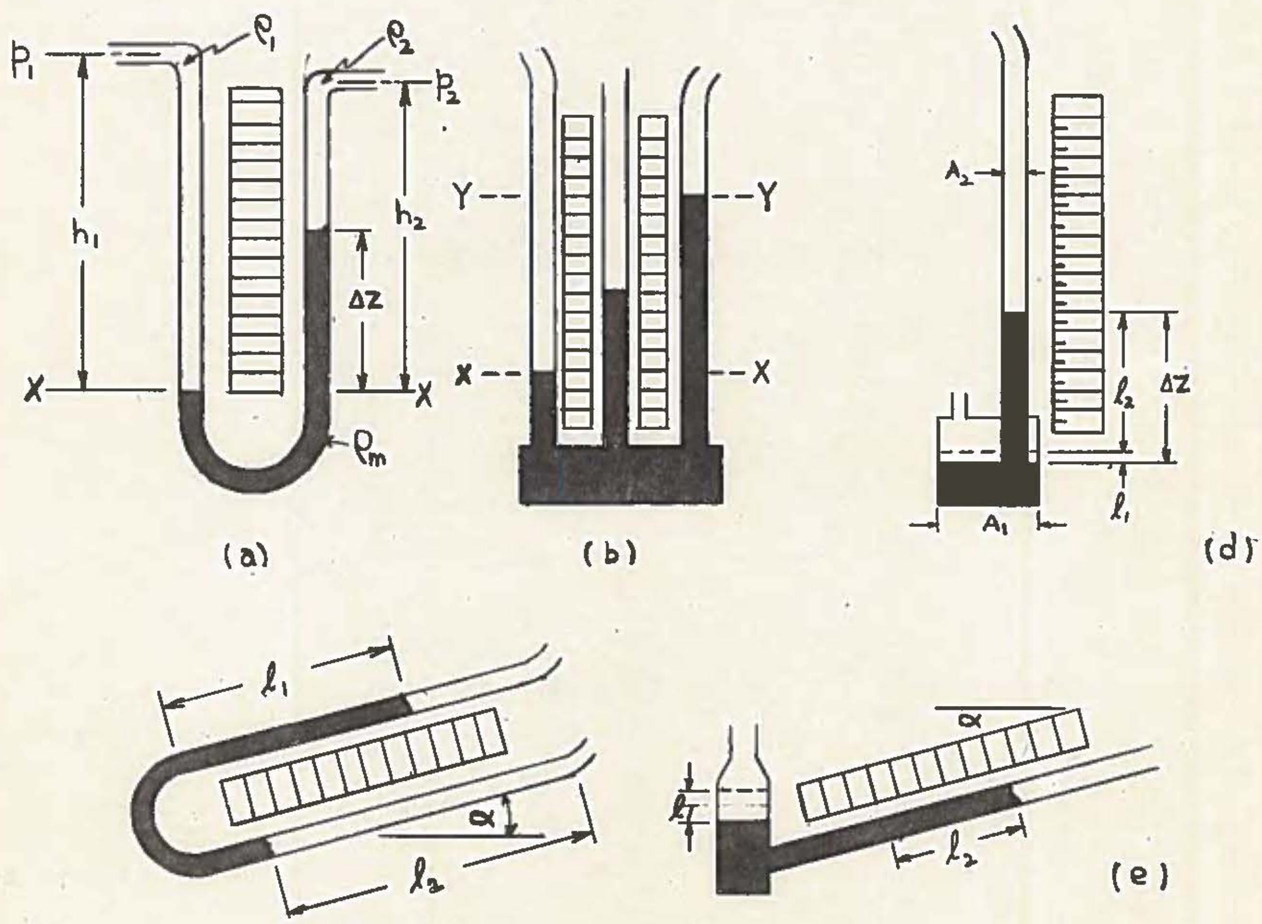

Fig. 2-3

The fluid indicated in arrangement (a) as having density ρ_m will be referred to as the manometric fluid and is invariably a liquid. An inverted arrangement may and must be employed if it is the fluid of least density.

Analysis of the data secured by such devices proceed primarily from Proposition (D) of art. 2-2, although surface-tension corrections of observed data may be required and are considered below.

Relevant analyses are as follows:

Arrangement (a). In this simple U-tube arrangement observe that in horizontal plane X-X the pressure in both legs of the tube must be the same if the manometric fluid is homogeneous and stagnate. Thus expressing (eq. 2-1) and equating these pressures

$$p_1 + g\rho_1 h_1 = p_2 + g\left[\rho_2(h_2 - \Delta z) + \rho_m \Delta z\right], \text{ in consistent units; or}$$

$$p_1 - p_2 = g\rho_m \Delta z + \rho_2(h_2 - \Delta z) - \rho_1 h_1{}^{*} \qquad (2\text{-}3)$$

The factor 1/32.17 will enter if lbf, lbm and foot units are employed.

If, as frequently may be the case, $\rho_1 = \rho_2 = \rho_{1,2}$

$$p_1 - p_2 = g\left[(\rho_m - \rho_{1,2})\Delta z + \rho_{1,2}(h_1 - h_2)\right]$$

$$= g\,\rho_{1,2}\left[(\rho_m/\rho_{1,2} - 1)\Delta z + (h_2 - h_1)\right] \qquad (2\text{-}3a)$$

$$\text{or} \quad = \gamma_{1,2}\left[(\gamma_m/\gamma_{1,2} - 1)\Delta z + (h_2 - h_1)\right] \qquad (2\text{-}3b)$$

Or if the fluid in the connections is a gas of negligible density relative to that of the manometric liquid (e.g., $\rho_{1,2}/\rho_m < .001$),

$$p_1 - p_2 = g\,\rho_m \Delta z\,. \qquad (2\text{-}3c)$$

Several noteworthy observation relating to the above are as follows:

(1) Writing Equation 2-3b in terms of ratios $p_1/\gamma_{1,2}$ and $p_2/\gamma_{1,2}$, these are seen to have the dimensions of force/area and (gravitational) force/volume, and thus the dimensional significance of length. In accordance with Equation 2-1b, either ratio would express also the vertical height of the column of fluid, of local sp. wt. γ, which in an open-ended tube would cause, or would be supported by,

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* - Recall that table 1-3 provides in effect the items g /32.17 for the conversions of "inches of mercury" and "feet of water" to pressures in lbf/sq. ft.

gauge pressure p at its base. Such a tube would be known as a piezometer tube, and not unnaturally the ratio is known as the pressure head corresponding to pressure p and sp. wt. γ . Ratios of the form p/ρ , of equation 2-3, have however the dimensions of force/area (FL^{-2}) divided by mass/volume (ML^{-3}), and thus the significance of (force x distance)/mass or energy per unit mass (FLM^{-1}, or $L^{2}T^{-2}$). This item becomes of much significance in subsequent analyses of flow phenomena. Although at or near the earth's surface, and if the non-consistent lbf and lbm units are used, the ratios p/γ and p/ρ become effectively equal numerically, the complete difference in their individual significances must be recognized.

Arrangement (b). This "gang" arrangement differs from the single U-tube only in its ability to measure readily several successive pressure differences. Analysis requires merely the recognition that in horizontal plane X-X the pressure is the same in all legs, or that in plane Y-Y it is the same in legs 2 and 3.

Arrangement (c). To give a greater linear displacement of the separation-surfaces, for a given vertical displacement, thereby enabling greater relative precision by measuring a greater length, the U-tube is frequently oriented with the legs at a moderate angle (^{o}C) with the horizontal. Their vertical displacement thus becomes $\sin\alpha$ times the linear. If the plane of both legs is at this angle, the displacement (l) may be measured directly. If their plane is, however, vertical, as in sketch (c), the relevent length is the sum of the displacements in each leg (i.e., l_1 l_2) from the positions of the separation-surfaces when p_1 = p_2 and when they are thus in the same horizontal plane.

It is evident that, for realizing the presumptive benefit of greater linear displacement, comparable precision in measuring α is necessary. Difficulties in exact location of a separation surface in an inclined tube may also be encountered.

Arrangement (d). A convenience of this arrangement is that it may readily be so devised that the position, along the vertical scale, of only the separation-surface in the tube needs be read. It will be recalled that in the similar mercurial barometer, with transparent-walled reservoir, this is enabled and caused by bringing the zero of the vertical scale to the plane of the mercury surface in the reservoir.

If greater pressures and desired ruggedness require a metallic reservoir wall and fixed scale, its graduations may still be made to indicate Δz directly by noting that $\Delta z = l_2 \;(1 + A_2/A_1)$, where A_2 and A_1 are respectively the transverse interior area of the tube and the like <u>net</u> area of the reservoir. That is; the fluid displacements on imposing a pressure difference must be such that $l_1A_1 = l_2A_2$, or $l_1 = l_2(A_2/A_1$, but $\Delta z = l_2 + l_1$. The zero point of this scale graduation is necessarily fixed at the plane of zero fluid displacement, when $p_1 = p_2$.

<u>Arrangement (e)</u>. This represents a common combination of arrangements (c) and (d). Provision of a fixed and direct-reading scale along the inclined tube would require graduations such that $\Delta z = l_2 \;(\sin\alpha + A_2/A_1)$, and proper positioning of the scale.

Referring to eq. 2-3, observe that the use of a manometric fluid such that $\rho_m/\rho_{1,2}$ only moderately exceeds unity will give a much increased surface displacement (Δz) for a given pressure difference, but also will require precise data on the individual densities.

A necessary consideration in these devices is the influence of surface tension in causing an additional elevation or depression of the fluid inside the tube. This "capillary action" occurs by virtue of the wetting of the tube (such as water in glass) or the pronounced non-wetting (such as mercury in glass) by the fluid and necessitates the use of a small corrective term when using certain types of barometers and manometers for pressure measurements. Figure II-4 illustrates a) a downward curved meniscus for non-wetting combination of tube and liquid and b) an upward curved meniscus where wetting occurs. The amount of the correction is the height of the rise or depression indicated as h in the figure. Since the sur-

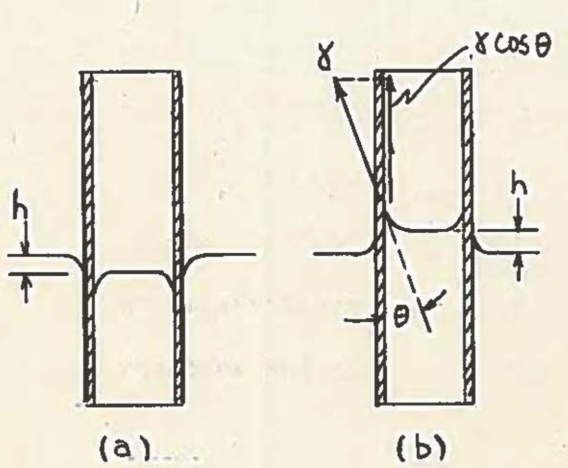


Figure II-4

face of the liquid makes some angle, θ, with the tube, the total vertical component of the surface tension force (equal and opposite to the force of adhesion between the tube and the liquid surface) is $(\gamma \cos \theta)\, 2\pi r$, where r is the radius of the tube. As a result of this force the liquid rises (or falls) in the tube until its weight just balances the force. Thus,

$$(\gamma \cos \theta)\, 2\pi r = \pi r^2 h w$$

or

$$h = \frac{2\, \gamma \cos \theta}{r w} \qquad \text{II-4}$$

For pure water and clean glass, the angle θ is zero, or the meniscus formed is effectively hemispherical. In a 0.2 inch diameter tube, the rise is about 0.2 inches. With mercury in the same glass tube, the depression is only about 0.03 inch.

The following examples illustrate various of the foregoing considerations.

Example II-1. In figure II-5 vessel A and the line connecting it with the U tube are filled with fluid a. Fluid b is used in the tube. Distance Z in the left leg, which is open to the atmosphere, is 6 inches, while distance Y is 2 ft. Determine the gage pressure at the center of vessel A, in lb/sq in, for the following cases:

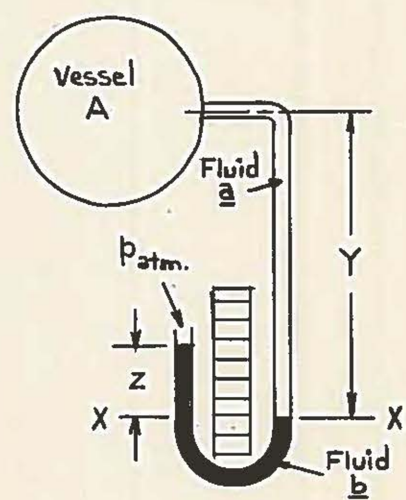


Figure II-5

a) Fluid a is air at a density of 0.075 lb/cu. ft., and b is water with specific gravity 1.0.

b) Fluid a is water and b is mercury with specific gravity of 13.6.

Solutions:- The pressure in the two legs at plane x-x is the same and is a function of p_A, p atm and the distances Y and Z. Thus

$$p_A + Y w_A = p_{atm.} + Z w_b \text{ , or } p_{A,gage} = Z w_b - Y w_a$$

For the cases a and b, and in the unit of psi

a) $p_{a,\ gage} = \dfrac{6/12 \times 62.4 - 2 \times 0.075}{144} = 0.217 - .001 = 0.216$ psi

b) $p_{a,\ gage} = \dfrac{6/12 \times 62.4 \times 13.6 - 2 \times 62.4}{144} = 2.95 - 0.87 = 2.08$ psi

Note that in case a the low density of the air in the connections makes its influence rather negligible. But in case b the density of the fluid in the connections is sufficient to make its accountability necessary. In addition the complete filling of the connecting tube is necessary for reliable computation.

Example II-2. - The left leg of the U-tube of figure II-5 is tipped forward to an angle of 15° with the horizontal. What is the gage pressure in vessel A if, in above case a, the linear distance along the tube between the two surfaces of liquid b is again 6 inches? Express the pressure in inches of water.

Solution. -

$$p_{a,\ gage} = 6 \times \sin 15^\circ - \frac{2 \times 12 \times 0.075}{62.4}$$

$$= 1.55 - 0.03$$

$$= 1.52 \text{ inches of water}$$

In this instance the relative influence of the air in the connection is not so negligible as in the previous example. Note also that an error of 1° in the angle of inclination would produce a relative error amounting to (sin 15°-sin 14°) /sin 15°, or 6.5 per cent.

Example II-3.- The pressure drop along a length of horizontal pipe through which water is flowing is determined by a differential manometer arrangement in which mercury is used as the manometric fluid (refer to Fig. II-6).

If the connections are completely filled with water, what is the pressure drop corresponding to a manometer reading (2) of 10 inches of mercury?

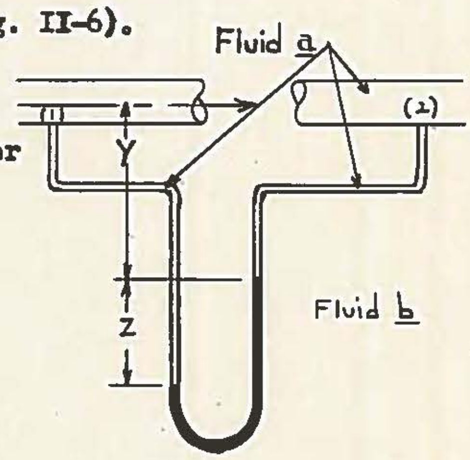


Figure II-6

Solution.- Equating pressures along plane x - x

$$p_1 + (Y+Z)w_a = p_2 + Yw_a + Zw_b$$

$$p_1 - p_2 = Z(w_b - w_a) = Zw_a\left(\frac{Sp.\ Gr._a}{Sp.\ Gr._b} - 1\right)$$

$$= \frac{10}{12} \times 62.4\left(\frac{13.6}{1.0} - 1\right)$$

$$= 655 \text{ lb/sq. ft}$$

Equivalents are $\frac{655}{144} = 4.55$ lb/sq in or $^{655}/62.4 = 10.5$ feet of water.

The above mercury-water combination is a frequent one, and the net multiplier 12.6 (versus 13.6) is thus often required.

2-6 Problems. -

2-1 Calculate the pressures in psi gage at a depth of 300 feet below the free surface of a body of fresh water at 60°F, and of a body of sea water of normal density (64 $\frac{\text{lb}}{\text{cu ft}}$ Ans. 129.9, 133.3 psig

2-2 The pressure at a depth of 10 feet below the surface in an open tank of liquid is 4.0 psig. What is the liquid density?

Ans. 57.6 $^{lb}/$cu. ft.

2-3 For the open tank of Figure II-7, containing fresh water, compute the following -

a) The pressure at the base, and the total hydrostatic force on it.

b) The hydrostatic force on the inclined side, and its vertical and horizontal components.

(72,200; 36,100; 62,500)

c) The aggregate of the vertical hydrostatic forces and the weight of the water within the container, comparing the two.

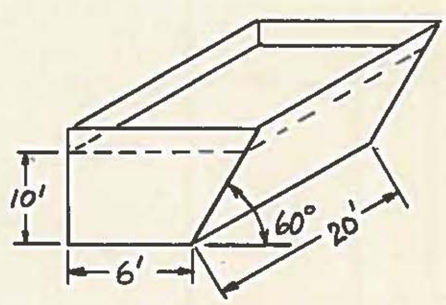


Figure II-7

2-4 The pressure in an oil line is 10 psig. To what height above the point of pressure measurement will the oil rise in a vertical tube which is open at the top if the specific gravity of the oil is 0.88? Express the pressure in the pipe both in "feet of oil" and in "feet of water".

Ans. 26.22 feet, 23.08 feet water

2-5 The pressure in a steam drum is measured by a gage located 15 feet below the drum. Because of the condensation the connecting tube becomes filled with water. What is the absolute pressure in the drum if the gage reads 23.5 psi and the atmospheric pressure is 30.0 inches of mercury. Ans. 31.7 psia.

2-6 What would be the absolute pressure in the drum of problem 5 if the gage indicated a vacuum of 10 inches of mercury?

Ans. 3.3 psia

2-7 For water at 170°F the vapor pressure is 5.99 psia. To what maximum height above an open supply tank might the atmospheric pressure lift water at this temperature to the suction of a pump if the atmospheric pressure is the standard pressure at sea level, the influence of dissolved gases in the water being neglected? What would be the height at an altitude of some 7,000 feet, where the atmospheric pressure is about 11.4 lb/in^2? Ans. 20.6 ft, 12.8 ft

2-8 By use of the relation 2-2 verify the pressure altitude data of problem 7 if at 7000 feet above sea level the temperature is normally about 500°F absolute (40°F). For the computation employ for the gas constant (R) for air the value of 53.3 ft. lb/lb, °F abs.

2-9 At the center line of the vessel of Figure II-8 the pressure is 25 psig. What will be the manometer reading (Z) in inches in the following cases?

a) Fluid <u>a</u> is water and completely fills the connecting pipe, fluid <u>b</u> is mercury, Y = 6 in. Ans. 51.4 inches

b) The fluids are as above, Y = 3 feet, but 4 inches of water has collected above the mercury column on the right.

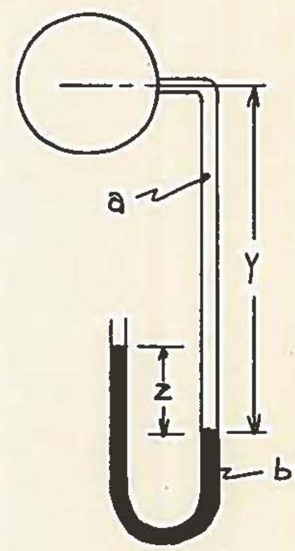


Figure II-8

c) Fluid <u>a</u> is oil of specific gravity 0.9, fluid <u>b</u> is mercury and Y = 3 feet. Ans. 53.3 inches

d) What would be the effect if air were entrapped in the upper bend of the connecting tube?

2-10 In the arrangement of Fig. II-9 air is introduced until, when all liquid has so been forced out of the vertical submerged tube, the mercury in the U-tube reaches and maintains the maximum reading of 12 inches. The liquid in the (closed) tank has a specific gravity of 0.8. What depth of liquid in the tank is so indicated? Ans. 17 ft.

$\frac{12 \times 34}{30 \times 0.8} = 17^{ft}$

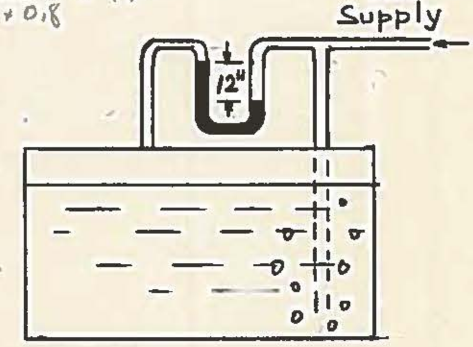


Figure II-9

2-11 The diameters of the inclined and vertical legs of the "draft gage" of Figure II-10 are 0.1" and 1.0" respectively. When pressure from the vessel A is imposed on the gage, the surface of the liquid in the inclined leg is displaced 6 linear inches along the tube from its "zero" position. The manometer liquid is kerosene, specific gravity 0.82, the right leg is open to the atmosphere and the left leg is connected to vessel A containing air. (density = .075 $\frac{\text{lb}}{\text{cu ft}}$) What is the gage pressure in the vessel, in inches of water?

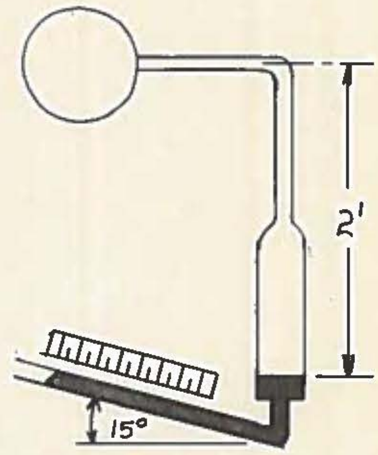


Figure II-10

Ans. 1.29" water

2-12 In the inverted arrangement of Figure II-11 fluid a is water of normal density and fluid b is oil of a specific gravity of 0.92. Distance z is 27 inches.
What is the value of $p_1 - p_2$ in lb/sq. ft. and in inches of water?
(2.16 inches of water)

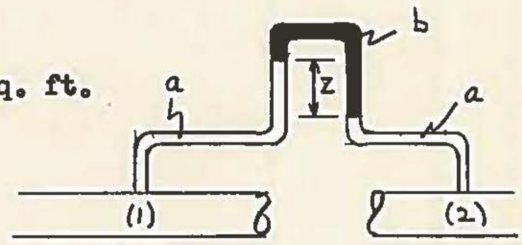


Figure II-11

2-13 If the arrangement in Fig. II-11 were that as required for the use of mercury as the manometric fluid, and the fluid in the pipe and connections were water, what would be the value of $p_1 - p_2$ if the manometer scale reading was again 27 inches?

2-14 In the arrangement of pipe line and several types of valves X , as shown in Fig. II-12, the fluid flowing through the line and filling the connecting tubes is oil (sp. gr. = 0.91) and the manometer liquid is mercury. Express in feet of oil the pressure drops $p_1 - p_2$, $p_2 - p_3$ and $p_3 - p_4$.

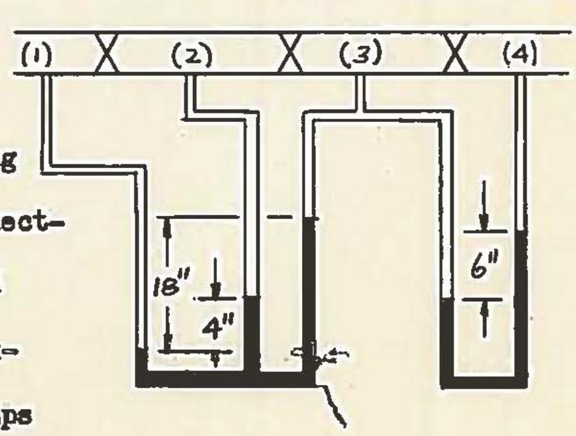


Fig. II-12

2-15 In the arrangement of larger piston, smaller piston and lever as shown in Figure II-13, what force will be required on the lever in order to exert a force of 10,000 lb. at the larger piston? Show that the work done by both pistons is the same.

Ans. 17.36 lb.

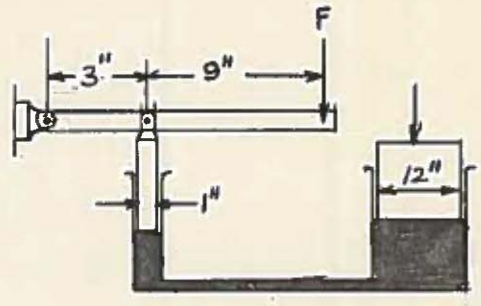


Figure II-13

CHAPTER 3

HYDROSTATIC FORCES and THEIR MOMENTS

3-1. Foreword: - It is frequently necessary for the engineer to evaluate forces acting on a submerged surface and the moment produced by such forces; both being caused by varying hydrostatic pressure in an overlying liquid column and perhaps also by the imposition of a supplementary pressure on the liquid surface. Typical instances include the force and moments on the face of a dam, the walls or bulkheads of a compartment, or the surfaces of a submerged or floating vessel.

3-2. Force Magnitude, Distribution and Components: - For the analysis of these it is initially simpler, and becomes sufficient for subsequent purposes, to consider the force on a plane surface, such as is shown in Fig. 3-1, which is oriented at angle α with the horizontal free surface of a liquid mass, and may actually or nominally extend to intersection with that surface at point O. The shape of a presumably important portion of the plane is represented and enclosed by line HIJK, when projected to a plane normal to the plane in question.

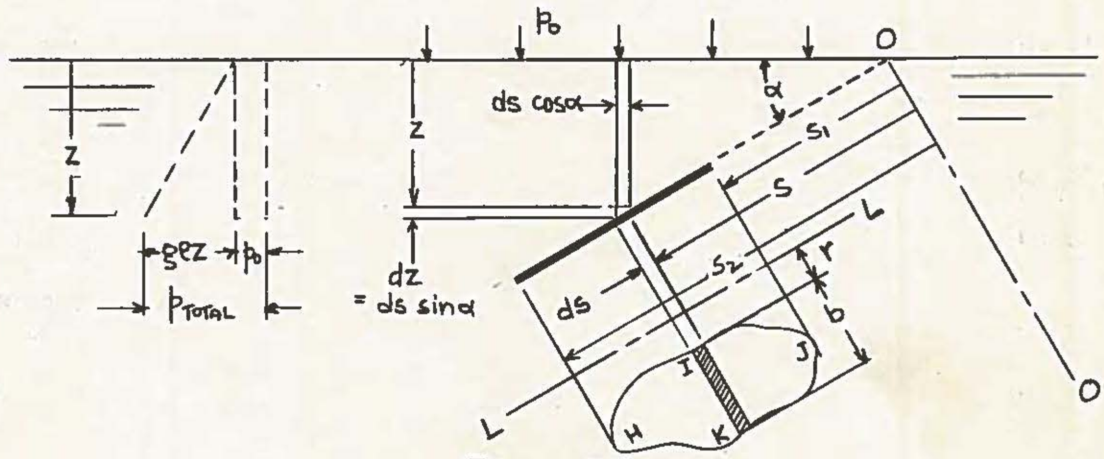


Figure 3-1

For generality a supplementary pressure p_o is taken to act on the liquid surface. The figure also indicates, to the left, the manner of vertical variation of the total pressure on the surface of the plane.

For the elemental horizontal segment of surface HIJK, of length b and breadth ds, and by Eq. 2-1a,

$$dF = p\ \text{total}\ dA = (p_0 + g\ \rho\ z)\ \ (b\ ds)^*$$

$$= (p_0 + g\ \rho\ s \sin\alpha\ )\ (b\ ds)$$

$$= p_0\ b\ ds + \left[g\ \rho\ \sin\alpha\right]/_2\ \ x\ bd(s^2)$$

and the total force on the surface becomes -

$$F = p_0 \int_{s_1}^{s_2} b ds + \frac{g \rho \sin\alpha}{2} \int_{s_1}^{s_2} b\, d(s^2) \qquad (3\text{-}1)$$

Note that $\int_{s_1}^{s_2} b\, ds$ is the area of surface HIJK.

If breadth b is expressible as a function of s the indicated integrations may be performed directly. For an irregularly-shaped surface, as in the figure, integration is readily done graphically, and with minor approximation, by plotting to careful scale values of b as ordinate and values of s and s^2 as abscissa. Fig. 3-2 illustrates.

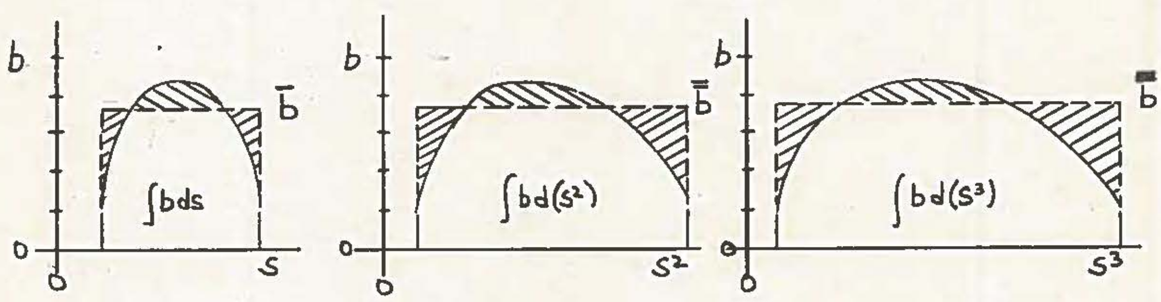


Figure 3-2

In the absence of planimeter mean values of b, such that

$$\bar{b}(s_2 - s_1) = \int_{s_1}^{s_2} b ds\ , \quad \text{and} \quad \bar{\bar{b}}(s_2^2 - s_1^2) = \int_{s_1}^{s_2} b\, d(s^2)$$

may be selected almost by visual equalization of the oppositely hatched areas indicated in the figure.**

As the ratio $\int_{s_1}^{s_2} b s ds \Big/ \int_{s_1}^{s_2} b ds$ is the distance (s_c) from 0 to the centroid of area HIJK the above relation may be written also in the form -

---

*For convenience following relations appear as for consistent force and mass units.

**More precision may evidently be otained by taking the average of the values of $\bar{b}$, or b, as found for each of several segments of equal linear breadth along the abscissas.

$$F = (p_o + g\,\rho\,s_c\,\sin\alpha)\int_{S_1}^{S_2} b\,ds. \qquad (3\text{-}1a)$$

Observe that, if the surface is simply submerged in and surrounded by the stagnate fluid, equal but opposing forces will act on both sides. But note also that an equal force would act on either side alone if the fluid were excluded from the other, and that for equilibrium an opposing equal and equivalently-distributed force must be provided extraneously. This may be done in part by a uniform pressure such as atmospheric, in which event the above p_o reduces in effect to the gauge pressure.

The horizontal and vertical components of (normal) force F may become of considerable individual concern and convenience. They develop as follows:

(a) Horizontal Component, F_h: - From Eq. 3-1

$$F_h = F\sin\alpha$$

$$= p_o\int_{s_1}^{s2} b\,ds\,\sin\alpha + g\rho\int_{s_1}^{s2} b\,(s\sin\alpha)\,(ds\sin\alpha)$$

$$= p_o\int_{z_1}^{z2} b\,dz + g\rho\int_{z_1}^{z_2} b\,z\,dz. \qquad (3\text{-}2)$$

But this is recognized as expressing also the force which would act on the projection of the surface in question to a vertical plane. Furthermore, it would do so even if angle α were variable; that is, if the surface were curved instead of plane.

(b) Vertical Component, F_v: - Similarly

$$F_v = F\cos\alpha$$

$$= p_o\int_{S_1}^{S_2} b\,ds\cos\alpha + g\rho\int_{S_1}^{S_2} b(s\sin\alpha)(ds\cos\alpha) \qquad (3\text{-}3)$$

But observe in this relation (1) that $\int b\,(\cos\alpha\;ds)$ expresses the area of the horizontal projection of surface HIJK, and does so whether that is plane or curved; and (2) that also as $(\sin\alpha)\,s = z$, the second integral represents the volume of the liquid overlying the surface. Thus

F_v = p_o x area of the horizontal projection of the surface + weight of the liquid which overlies the surface

Again observe that these manners of force evaluation continue to be valid even if the surface in question were a portion of the wall of a container from which liquid is in fact excluded, but its submergence in liquid environs is enforced.

3-3. <u>Hydrostatic Moment; Center of Pressure</u>: - The moment about some significant axis produced by a hydrostatic force and any uniform supplementary pressure-force, acting on a submerged surface may well become quite as essential as the determination of the forces themselves. Its consideration again proceeds more easily by initial attention to the moment of a plane surface, such as HIJK in Fig. 3-1, and with reference to the horizontal axis through point <u>O</u>, or moment $\underline{M_o}$.

For differential area <u>b ds</u> this moment is -

$$dM_o = (p_o + g\rho z)(b\,ds)\,s$$
$$= (p_o + g\rho s \sin\alpha)(b\,ds)\,s;$$

and for surface HIJK becomes

$$M_o = p_o \int_{s_1}^{s_2} b s\,ds + (g\rho \sin\alpha)\int_{s_1}^{s_2} b s^2\,ds \quad \text{or} \qquad (3\text{-}4)$$
$$= \frac{p_o}{2}\int_{s_1}^{s_2} b\,d(s^2) + \frac{g\rho\sin\alpha}{3}\int_{s_1}^{s_2} b\,d(s^3)$$

Recall that Fig. 3-2 represented both the first integral and a suitable mean value of $\bar{\bar{b}}$; note also that the figure indicates the second integral and a mean value of $\bar{\bar{\bar{b}}}$ such that $\bar{\bar{\bar{b}}}\,(s_2^{\,3} - s_1^{\,3}) = b\,d(s^3)$. The integral $\int b\,s^2\,ds$ will be recognized as the second moment, or "moment of inertia" (I_o), of area HIJK about axis O-O. Then

$$M_o = p_o s_c A + g\rho \sin\alpha\, I_o$$

In this connection note that just as the ratio $\int b\,s\,ds / \int b\,ds$ defines the distance s_c from O to the centroid of the area, the ratio $\int (b\,s^2\,ds) / \int b\,s\,ds$, that between the moment of the hydrostatic force and that force, is a distance from O to such point that the force, <u>if concentrated there</u>, WOULD PRODUCE THE SAME MOMENT. This point is known as the <u>center of</u> (hydrostatic) <u>pressure</u>, (cp). Introducing these items in Eq. 3-4,

$$M_o = p_o s_c \int_{s_1}^{s_2} b\,ds + g\rho\sin\alpha\, s_{cp}\int_{s_1}^{s_2} b s\,ds$$
$$= (p_o + g\rho\sin\alpha\, s_{cp})\, s_c \int_{s_1}^{s_2} b\,ds \qquad (3\text{-}4a)$$

Relative to a lateral axis (L-L) in the plane of the surface but normal to O-O, and again considering differential areas b ds, the moment about the axis is expressible as -

$$M_L = p_0\int_{S_1}^{S_2}\left(r+\frac{b}{2}\right)b\,ds + g\rho\sin\alpha\int_{S_1}^{S_2} s\left(r+\frac{b}{2}\right)b\,ds$$
$$= p_0\int_{S_1}^{S_2}\left(r+\frac{b}{2}\right)b\,ds + \frac{g\rho\sin\alpha}{2}\int_{S_1}^{S_2}\left(r+\frac{b}{2}\right)b\,d(s^2) \qquad (3\text{-}5)$$

These two integrals, as they relate to the surface of Fig. 3-1, are indicated graphically in Fig. 3-3.

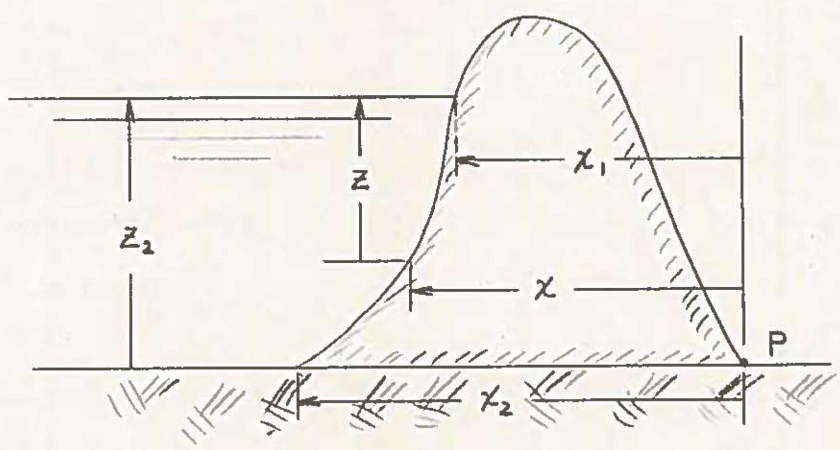


Figure 3-3

It may frequently be desired to determine the moments about some horizontal axis which parallels but is not in the plane of a surface. Fig. 3-4 illustrates for the case of a common structure, the <u>gravity dam.</u> In this situation it becomes more convenient to consider individually the horizontal and vertical components of the forces on the surface, perhaps per unit breadth (i. e., F/b), and their individual moments. Paralleling the relations of equations 3-2, 3-3, and 3-4, but omitting terms in p_0 as a supplementary surface pressure would rarely be operative in this situation, the individual forces and their moments are as follows -

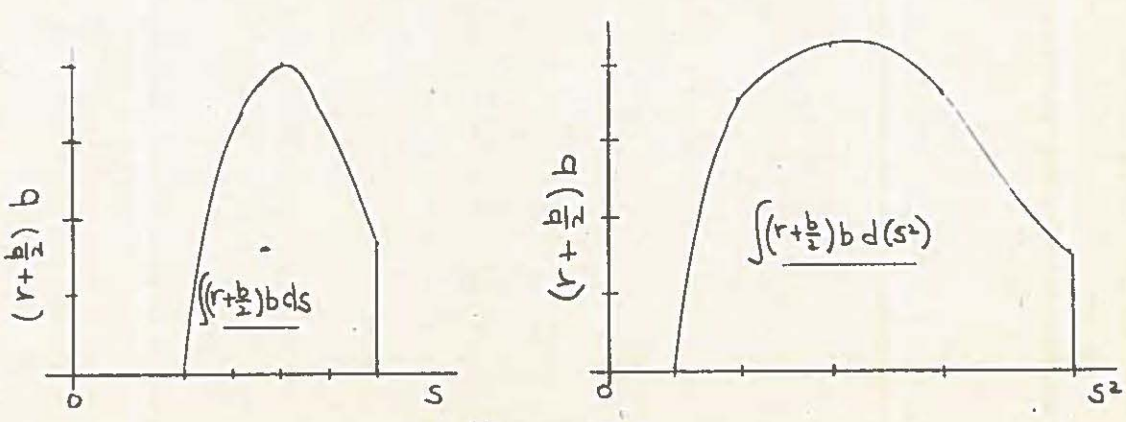


Figure 3-4

For the horizontal component, per unit length of dam,

$$F_h = g\rho\int_0^{z_2} z\,dz = g\rho z_2^2/2 \quad \text{and}$$

$$M_{h,P} = g\rho\int_0^{z_2}(z_2 - z)\,z\,dz = g\rho z_2^3/6 \quad \text{and clockwise} \qquad (3\text{-}6)$$

Observe that force F_h acts both to displace the structure horizontally and to capsize it. Although in fact a distributed force, it may for direct moment computation be regarded as a concentrated one acting at distance $z_2/3$ above P.

For the vertical component, per unit length

$$F_v = g\rho\int_{x_1}^{x_2} z\,dx \quad \text{and}$$

$$M_v = g\rho\int_{x_1}^{x_2} z\,x\,dx = \frac{g\rho}{2}\int_{x_1}^{x_2} z\,d(x^2) \qquad (3\text{-}7)$$

and counterclockwise. Graphical integration would again be convenient.

Observe that F_v is the weight of the water overlying the <u>face</u> of the dam, and its moment is a stabilizing one. The sum of this moment plus that due to the weight of the material forming the structure must exceed sufficiently the sum of the capsizing horizontal moment and of any upward force and consequent capsizing moment due to possible liquid penetration beneath the under-surface of the structure.

The following example illustrates various of the foregoing considerations.

<u>Example 3-1.</u> For the trapezoidal surface in the wall of the vessel of Fig. 3-5 determine the following items using graphical methods. The fluid is fresh water of a density of 62.4 lbm or 1.94 slugs/cu.ft.

(a) The hydrostatic force, normal to the surface, and its moment about horizontal axis 0-0.

(b) The location, relative to that axis, of the center of pressure, and that of the centroid of the surface.

(c) The horizontal and vertical components of the hydrostatic force.

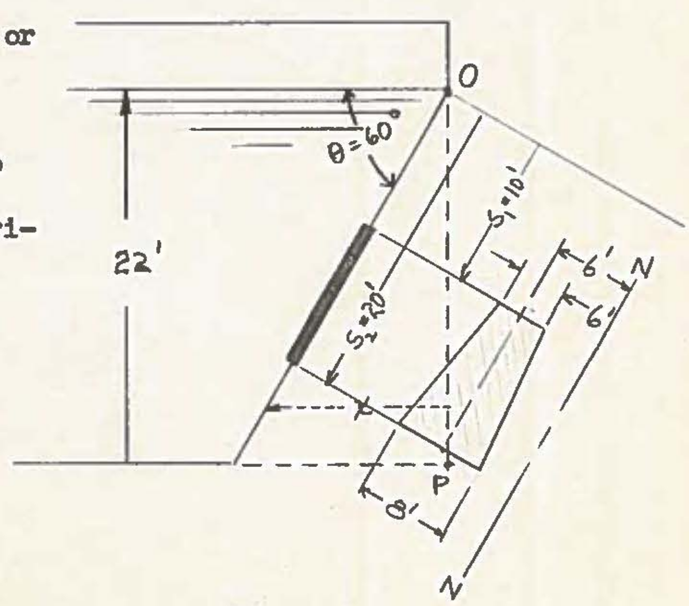


Figure 3-5

(d) The resultant moment, due to these force components, about a horizontal axis through point P (Fig. 3-5).

(e) The (hydrostatic) moment about axis N-N.

<u>Solution:</u> Preliminary items:- $g\rho = 62.4$ lbf/cu.ft; $\sin\theta = .866$; $g\rho\,\sin\theta = 54.04$; $\sin^2\theta = .75$; $\sin^3\theta = .65$; $\cos\theta = .50$

s	10	12	14	16	18	20 ft
s^2	100	144	196	256	324	400 sq.ft.
s^3	1000	1728	2744	4096	5832	8000 cu.ft.
b	6.0	6.4	6.8	7.2	7.6	8.0 ft.

The graphs of Fig. 3-6 exhibit the associated variation of b to abscissas of s, s^2 and s^3, corresponding values of $\bar{b}$, $\bar{\bar{b}}$, and $\bar{\bar{\bar{b}}}$ as found by area equalizations, and consequent values of $\int b\,ds$, $\int b\,d(s^2)$ and $\int b\,d(s^3)$.

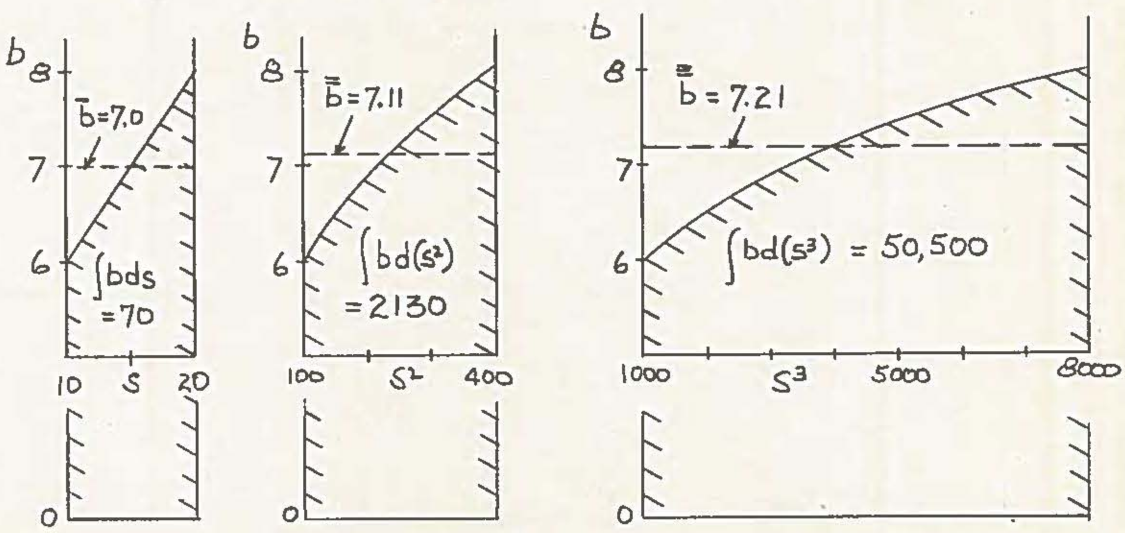


Figure 3-6

(a) $F_{hyd} = (62.4/2)\ \sin\ 60^\circ \int_{s_1}^{s_2} b\ d(s^2) = 27.02 \times 2130 = 57{,}550$ lbf

$M_{hyd} = (62.4/3)\ \sin\ 60^\circ \int_{s_1}^{s_2} b\ d(s^3) = 18.01 \times 50{,}000 =$ 909,500 lbf. ft.

(b) $S_{cp} = 909{,}500/57{,}550 = 15.80$ ft. As the centroid location is determined by the ratio of the moment of the area to that area,

$$s_c = \left[\int b\ d(s^2)/2\right] / \int b\ ds = 2130/(2 \times 70) = 15.21 \text{ ft.}$$

(c) $F_{h'hyd} = g\rho \int_{z_1}^{z_2} b\, z\, dz = (g\rho/2) \sin^2 60° \int_{S_1}^{S_2} b\, d(s^2)$

$= 62.4/2) \times .75 \times 2130 = 49{,}800$ lbf

$F_{v'hyd} = g\rho \sin 60° \cos 60° \int_{S_1}^{S_2} b\, d(s^2)/2$

$= (62.4/2) \times .866 \times .50 \times 2130 = 28{,}800$ lbf

(d) Regarding counter-clockwise moments about P as positive, recalling that $z = S \sin\theta$ and noting that for the surface in question the x of Eq. 3-7 equals $s \cos\theta$;

$$M_{hyd}, \text{ about } P, = -g\rho \int_{z_1}^{z_2} b\, z\,(22-s)\, dz + g\rho \int_{z_1}^{z_2} b\, z\, x\, dx$$

$$= -g\rho \left\{ \int_{z_1}^{z_2} \left[22\, b\, d(z^2)/2 - b\, d(z^3)/3\right] - \int_{S_1}^{S_2} b\,(\sin\theta)\,(\cos^2\theta) s^2\, ds \right\}$$

$$= -g\rho \left\{ \int_{S_1}^{S_2} \left[11 \sin^2\theta\, b\, d(s^2) - (\sin^3\theta/3) b\, d(s^3)\right] - (\sin\theta \cos^2\theta/3) \int_{S_1}^{S_2} b\, d(s^3) \right\}$$

$$= -62.4 \left\{ \left[11 \times .75 \times 2130 - (.65/3)50{,}500\right] - .866 \times .50^2 \times 50{,}500/3 \right\}$$

$$= -62.4(6{,}650 - 3{,}650) = -187{,}000 \text{ lbf.ft.}$$

Refer to problem 3-9 for an alternate method of solution.

(e) As the pressure is constant along any line normal to an axis such as N-N, and that axis is at a constant distance of 6.0 ft. from the center-line of the surface,

$$M_{hyd}, \text{ about N-N} = 6.0 \times g\rho \int_{S_1}^{S_2} z\, b\, ds$$

$$= \left[(6.0 \times 62.4 \times \sin\theta)/2\right] \int_{S_1}^{S_2} b\, d(s^2)$$

$$= 162 \times 2130 = 346{,}000 \text{ lbf. ft.}$$

Example 3-2. Again for the surface in the wall of the vessel of Fig. 3-5, but now covered and with an air pressure of 10 psi at the liquid surface, determine the above forces and moments as due to this pressure; also the sum of these and those due to the hydrostatic pressure.

Solution.

(a) $F_p = 1440 \int_{S_1}^{S_2} b\, ds = 1440 \times 70 = 100{,}800$ lbf

$\Sigma F = 57{,}550 + 100{,}800 = 158{,}400$ lbf

$M_p = (1440/2) \int_{S_1}^{S_2} b\, d(s^2) = 720 \times 2130 = 1{,}534{,}000$ lbf. ft.

$\Sigma M = 909{,}500 + 1{,}534{,}000 = 2{,}444{,}000$ lbf. ft.

(b) To an "equivalent center of pressure", attributable to the aggregate moments and forces, s = 2,444,000/158,400 = 15.43 ft.

(c) $F_{hor,\ p} = F_p \sin \theta = 100{,}800 \times .866 = 87{,}300$ lbf

$F_{vert,p} = F_p \cos \theta = 100{,}800 \times .50 = 50{,}400$ lbf

(d) $M_{p,}$ about P, $= - 87{,}300\ (22 - 15.22 \sin \theta) + 50{,}400 \times 15.22 \cos \theta$

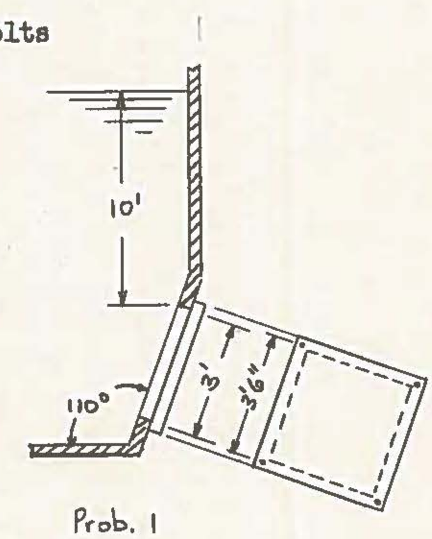
$= - 778{,}000 + 383{,}000 = - 395{,}000$ lbf. ft.

$\sum M = - (187{,}000 + 395{,}000) = - 582{,}000$ lbf. ft.

(e) $M_{p,}$ about N-N, $= 6.0 \times 1440 \int_{s_1}^{s_2} b\, ds = 6.0 \times 1440 \times 70 = 605{,}000$ lbf. ft.

$\sum M = 346{,}000 + 605{,}000 = 951{,}000$ lbf. ft.

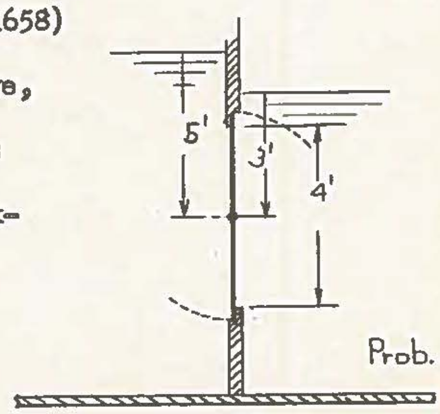
3-4. Problems



Prob. 1

3-1. Determine the stresses that four bolts should be capable of withstanding for retaining in position the plate shown in the accompanying figure. The opening covered is 3 ft by 3 ft, with the upper edge 10 ft (vertically) below the surface of the water in the open tank. The bolts are at the corners, on 3' - 6" centers.

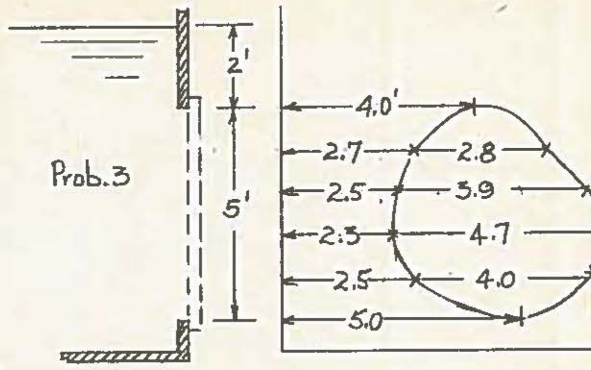
Ans. (upper bolts, 1543 lb each; lower 1658)



Prob. 2

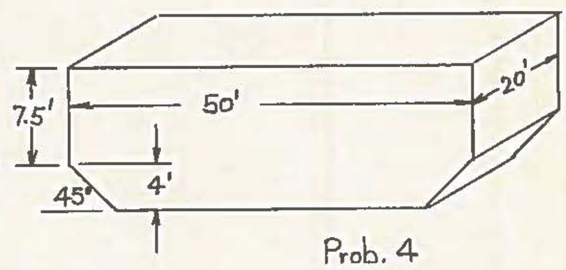
3-2. A circular "butterfly" type of valve, hinged on the horizontal axis through its centroid, is installed in a vertical bulkhead separating two water reservoirs, as shown in the figure. The radius of the opening which the valve closes is 2 ft. Will the valve tend to remain closed or to open when the water levels in the reservoirs are as indicated, and what initial torque will correspondingly be required to open or to close it?

3-3. The location and shape of a patched opening, in a vertical bulkhead forming a side of an open compartment which contains oil of specific gravity of 0.90, is shown in the figure. What will be the force on the plate covering the opening and the location of its center of pressure, the latter both vertically and laterally?



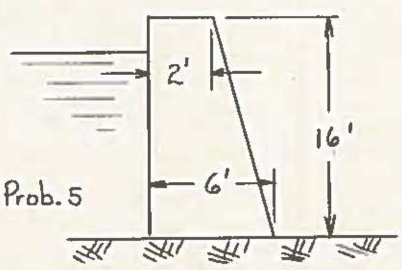
Prob. 3

3-4. The barge shown in the figure has a draft of 3 ft in sea water when light, or unloaded, and of 6 ft when loaded with cargo. What is its weight light, and the weight of the cargo when so loaded? Ans. 172,800; 191,000.



Prob. 4

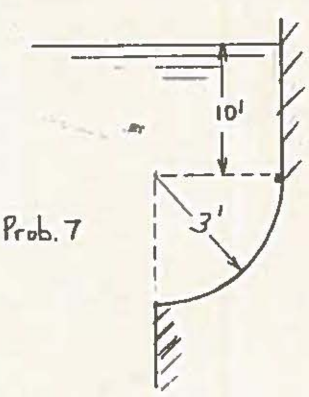
3-5. At what depth of water would the concrete gravity dam (w = 150 lb/cu ft) become clearly unsafe?
Ans. 15.2 ft.



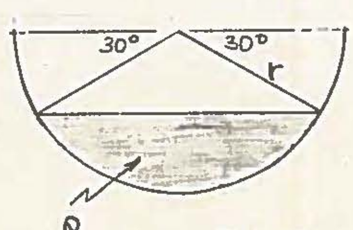
Prob. 5

3-6. What is the water level, measured from the top of the gate of problem 3-1 when its center of gravity is 2 inches above the center of pressure? Ans. 2.82 ft.

3-7. Show by integrating the hydrostatic pressure forces along the quadrant shaped gate of Fig. 3-12 that the vertical components of these forces are equal to the weight of water above the gate, and that the horizontal components are equal to the hydrostatic force on a projection of the gate on a vertical plane.

Prob. 7


3-8. A cylindrical water conduit with semi-circular cross section is partially filled with water as shown in Fig. 3-13. Compute the maximum bending moment in the conduit.



Prob. 8

3-9. Check part d of example 3-1 by taking the moment of the hydrostatic force, or its vertical and horizontal components, about a horizontal axis through P.

CHAPTER 4

BUOYANCY, AND STATIC STABILITY

4-1. Foreword. In the preceding chapters we noted that the hydrostatic forces on the two sides of a submerged surface are equal but opposing if the surface is in static equilibrium; that both are a function of the specific weight of the fluid and of the depth; and that the same hydrostatic force would act on either side alone even if the fluid were excluded from access to the other. As a consequence of the last consideration; there exists on the submerged under-surface of any vessel, from which the environmental fluid is excluded a net upward or buoyant force which opposes the downward gravitational force due to the mass of the vessel and its contents. Furthermore the buoyant and the gravitational may be so related that their couple will act to maintain the vessel in a desired posture (i.e. orientation with respect to the vertical.)

The terms buoyancy and (static) stability express these two actions. Related analyses, sufficient for present purposes, are provided in the following.

4-2. Buoyancy. For recognizing more specifically the nature and magnitude of the buoyant force on a submerged vessel, or any body from the interior of which the surrounding fluid is excluded, refer to sketch (a) of Fig. 4-1. There, in accordance with Eq. 3-1, the downward force produced on the upper surface of the body if at depth z and by an overlying tube of fluid of mean density ρ_f and transverse area dA, and the upward force on a corresponding lower surface if at submergence $z + \Delta z$, will be so related that

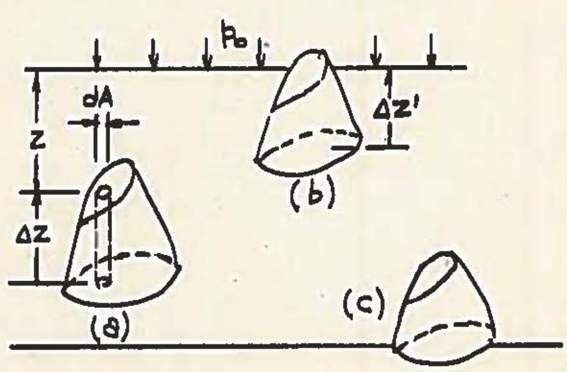


Figure 4-1

$$dF_b = -g\rho_f z \; dA + g\rho_f (z + \Delta z)\, dA$$
$$= g\rho_f \; \Delta z \; dA,$$

where dF_b represents a net buoyant force. Downward forces will be regarded as negative in sense, and upward ones as positive.

Denoting by ρ_b the mean density of the material forming the body, its weight (dF_g) equals $g\ \rho_b\ \Delta z\ dA$. If this downward gravitational force and the above upward buoyant force differ, a resultant accelerative force (dF_a) will evidently act on the element of the body in the amount

$$dF_a, = dF_b - dFg, = g\ \rho_f\ \Delta z\ dA - g\rho_b z\, dA$$
$$= (\rho_f - \rho_b)\ g\ \Delta z\ dA. \qquad (4\text{-}1)$$

Indicating by prefix $\sum$ the aggregate forces acting on the composite of all elements forming the body,

$$F_a = \sum g\ \rho_f\ \Delta z\ dA - \sum g\ \rho_b\ \Delta z\ dA$$
$$= g \sum (\rho_f - \rho_b)\ \Delta z\ dA. \qquad (4\text{-}2)$$

Interpreting this relation in several significant manners,

(a) The item $\sum (g\ \rho_f\ \Delta z\ dA)$ represents also the weight of the fluid which is displaced by the intrusion of the vessel in its fluid environs, and $\sum (g\ \rho_b\ \Delta z\ dA)$ that of the vessel and contents. Tendency to upward or downward motion of the vessel thus depends directly on the deficit or excess of its weight relative to that of the fluid it displaces. The tendency disappears only if the two are equal.

(b) If the weight of the body is less than that of the fluid it displaces the tendency to upward travel will however decrease if it begins to emerge through a free surface of the fluid, and so to displace a lessening fluid volume. It will become zero, and the vessel will float at this surface, when the emergence is such that the ultimately displaced weight of fluid equals that of the body. This is the situation indicated in sketch (b) of Fig. 4-1.

These obervations may be recalled as expressions of the long-familiar Principle of Archimedes.

(c) If the weight of vessel and contents is the greater, downward motion will continue unless the buoyant force is sufficiently supplemented by contact with some unyielding material. A related situation is one in which the net downward force causes at least a partial embeding of the under-surface of the vessel in a plastic bottom, with accompanying extrusion of the liquid from access to that surface.

The bottom may be so impermeable as to impede any subsequent return of the liquid to contact with the under-surface of the body. In the extreme case of its full embeding, as indicated in sketch (c), the absence of the normal hydrostatic and buoyant force on that surface will require, for freeing the vessel from the bottom in any salvage operation, an initial upward pull equal to the aggregate of the weight of it and its contents plus that of the entire column of liquid which overlies it. This may be quite formidable.

(d) In a case such as that of a closed and gas-filled but elastic-walled container, such as a balloon, ascending in a gaseous medium such as the atmosphere, there will be progressive and concurrent changes (art. 2-4) both of the density of the environmental fluid and of the volume and consequent mean density of container and contents. Various consequences are possible. One may be an ultimate arrival at equality of densities and vertical equilibrium.

4-3. Stability Requirements: - Buoyancy provision alone will rarely be sufficient for practical purposes, whether for a submerged vessel or one floating at a liquid surface. Instead it must be accompanied by provisions such that the vessel will automatically assume a suitable posture; i.e., it shall possess stability. Such stability is provided if, on departure from the desired orientation, the upward buoyant force and downward gravitational force produce a restoring or righting moment through which the vessel may regain the proper posture.

A vessel which so tends to resist upsetting is said to be stable; one which exhibits no tendency either to depart from or return to any posture in which it may chance to be is in neutral equilibrium; and one which tends to depart from a desired posture is said to be unstable.

For more specific analysis of the conditions required for stability, and although the hydrostatic and gravity forces are in fact distributed ones, it is convenient and suitable to consider equivalent single forces, which may be regarded as acting through single points and which would furnish the same moment as would the distributed forces. For the gravitational force this point is clearly the center of gravity of the vessel and contents, denoted in Fig. 4-2 by symbol G.

From the foregoing, for the buoyant force this point is the center of gravity of the fluid which, if not displaced by the vessel, would occupy that space. The point is known either as the center of displacement or the center of buoyancy, and denoted by symbol B.

It is necessary to distinguish between stability requirements for a body which is submerged in an environmental fluid, and one which is floating at the surface of a liquid.

(A) Submerged Bodies. For a wholly submerged body in which, by chance, the centers of gravity and of buoyancy coincide, the situation is one of neutral equilibrium, as no righting moment may result from departure from any initial posture. For positive stability it is necessary that, as illustrated in Fig. 4-2, the center of gravity be below the center of buoyancy when in the desired orientation. If in this situation the body is given an angular displacement, θ, a righting moment comes into effect in the amount F_g $(= F_b)$ $(GB \sin \theta)$, where GB denotes the linear distance between G and B.

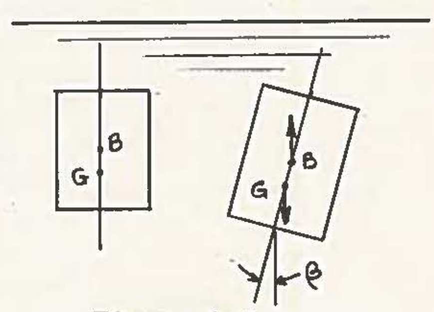


Figure 4-2

(B) Floating Bodies. For bodies such as a homogeneous sphere or cylinder, with G thus at the geometric center, the condition is again one of neutral equilibrium. Observe that for such a geometric form the location of B is independent of the orientation of the body, depending only on the relative densities of body and liquid and the consequent amount of emergence of the body above the liquid surface, but its location may not be above the center.

By an eccentric location of G, however, stability is obtained in the orientation for which line GB is vertical

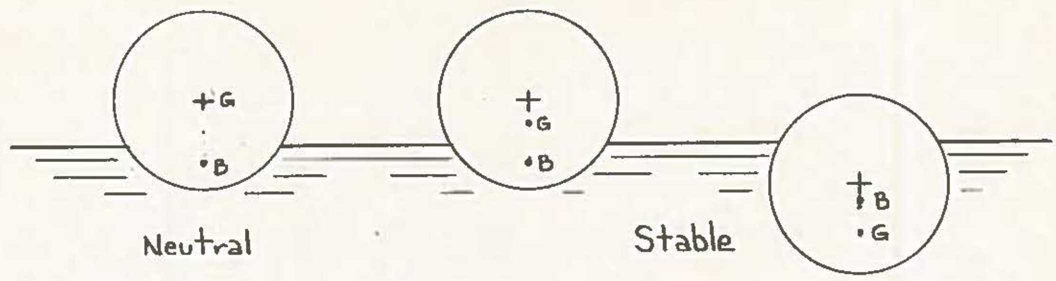


Figure 4-3

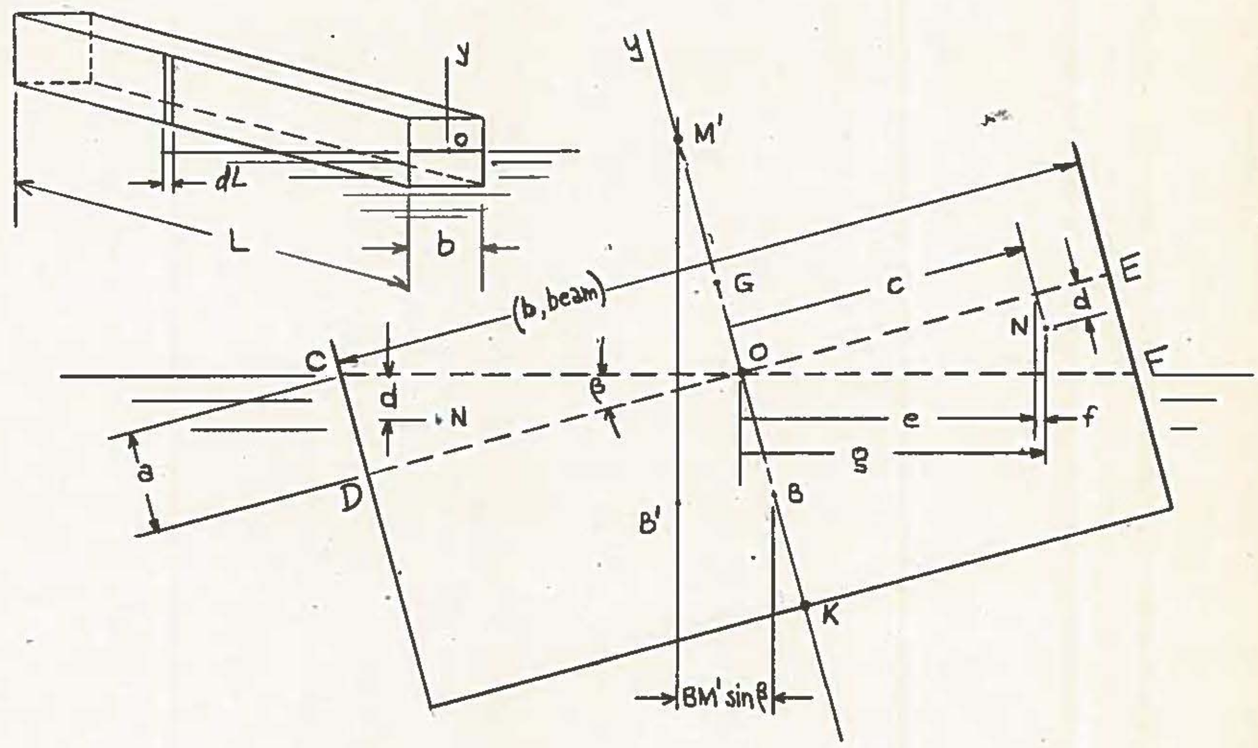
and G is at lowest position. Fig. 4-3 indicates the neutral condition and two stable ones, in the first of which it is notable that G is above B.

For the utilitarian surface vessel, such as the conventional ship, more complex considerations enter and are indicated in the following article.

4-4. Stability Analysis, Surface Vessels. The form of the submerged portion, or underbody, of the conventional surface ship generally approaches prismatic, with transverse crosssections in the midsection of the ship still more closely approaching rectangular. Such forms will be shown to permit construction and loadings for which stability is obtainable even with G located appreciably above B, with attendant advantages in flexibility and convenience. Stability over a wide range of inclination is still obtainable if the inclination is accompanied by a sufficient shift of the center of buoyancy to positions such that adequate righting moments persist.

Consider initially a simple, symmetrical prismatic form, such as that of the familiar barge shown in various views in Fig. 4-4. By action of wind or wave it has been inclined or "heeled" through angle θ . With reference first to any transverse lamina of thickness dl, by such inclination a triangular element or wedge GDO evidently becomes submerged and also a wedge OEF is caused to emerge from the surrounding water. As the weight of the vessel and the volume of the water it thus displaces is not modified by the inclination, the areas of the two wedges must be the same. If near the water line, the sides of the vessel are parallel to axis y-y, the wedges must have identical shapes as well.

Trigonometric relations pertinent to the area and location of the centroid (N) of the wedges are indicated in the figure. Evidently



$a = (b/2) \tan\beta$	$f = d \sin\beta$
$c = b/3$	$= (b/6) \tan\beta \sin\beta$
$d = (b/6) \tan\beta$	$g = e + f$
$e = (b/3) \cos\beta$	$= (b/3)\cos\beta\left[1 + (\tan^2\beta)/2\right]$

Wedge areas (each) $= (b^2/8) \tan\beta$

each contributes equally and cumulatively to a (leftward) shift of the center of bouyancy to a point B! A more important item, however, is the location of the intersection of the line of action of the bouyant force (F_b, through B') with axis y-y, at a point known as the pre metacenter corresponding to the existing inclination and identified as point M' in the figure. The distance of this point from the center of bouyancy prior to inclination will be denoted as BM'

By shift of the bouyancy center, to B', a moment is produced which equals the sum of the moments of the individual wedges about B. However, as these form a couple the magnitude of which about O is the same as that about B, from the quantities noted in Fig. 4-4 it may be expressed as $2 A_w (b/3)\cos\beta \left[(1+(\tan^2\beta)/2\right]$ where $A_w = (b^2/8)\tan\beta$. But this moment may be expressed also as $A(BM' \sin\beta)$, where A = total and unchanged under-water area.

Thus $\quad A(BM' \sin\beta) = (b^3/12)\sin\beta\left[1 + (\tan^2\beta)/2\right]$ (4-3)

and $\quad BM' = \left[1 + (\tan^2\beta)/2\right](b^3/12\,A)$* (4-4a)

and further, for a vessel of varying beam (b),

$$\overline{BM'} = \left(1 + \frac{\tan^2\beta}{2}\right)\frac{\int_0^L b^3\,dl/12}{\int_0^L A\,dl} \qquad (4\text{-}4b)$$

where $\overline{BM'}$ is the representative mean magnitude of BM' as it would pertain to the entire length of vessel (L) at the water-surface plane, or water-plane.

Finally, as the inclination approaches zero,

$$\overline{BM} = \int_0^L (b^3 dl/12)\Big/\int_0^L A\,dl \qquad (4\text{-}4c)$$

The length $\overline{BM}$ is evidently the minimum magnitude attainable by $\overline{BM'}$ and is known as the metacentric radius, point M being known as the (true) metacenter for the vessel when displacing liquid of volume $\int_0^L A\,dl$.

A graph indicating the growth of the function $\left[1 + (\tan^2\beta)/2\right]$ appears in Fig. 4-5. Corresponding values of BM', as expressed By Eq. 4-4b, are however valid only so long as the sides of the vessel are parallel to y-y. For smaller boats and sailing vessels outwardly-sloping sides are characteristic and contribute to a greater rate of increase of BM' with β. But the differing shapes of the emerging and submerging wedges involve more bothersome procedures for locating B'.

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*By writing suitable moments about O it may be seen further that the migration of B to B' ~~has~~ a component normal to y-y equalling (2c) × (A_w/A), or $(b^3/12A)\tan\beta$, and one upwardly parallel of (2d) × (A_w/A), or $(b^3/24A)\tan^2\beta$.

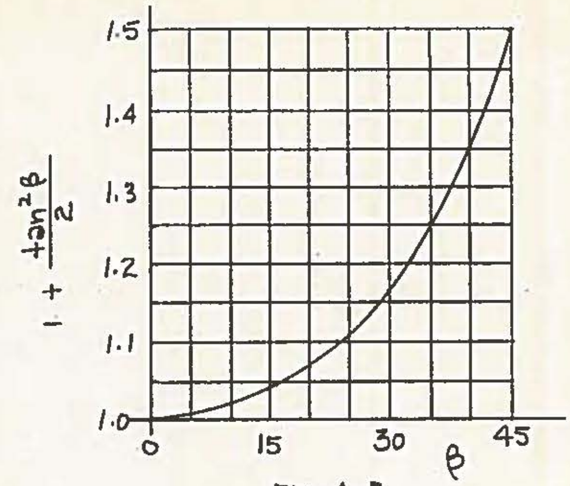

Fig. 4-5

The integrals $\int_0^L b\, dl/12$ and $\int_0^L A\, dl$ may evidently be determined from graphs of b^3 and of A, vs l as the abscissa. As the first integral expresses the second area moment or moment of inertia (I) of the water plane, when y-y is vertical, and here regarded as relative to its longitudinal axis, and the second integral expresses the volume (V) of liquid displaced by the vessel, Eq. 4-4c frequently appears as

$$\overline{BM} = I/V. \qquad (4\text{-}4d)$$

The second moment of the water plane relative to the transverse axis through its centroid determines similarly the vertical location of a longitudinal metacenter.

The major significance and utility of the metacenter, and also of the prometacenters, is that the location of G relative to these determines more directly the stability of a surface vessel than does its location relative to B, or B'. More specifically, it is apparent from Fig. 4-4 that an orientation of the vessel and a relative location of G such that it is merely (vertically) below M provides a stable situation. On any departure from this orientation a righting moment is introduced in the amount.

$$\text{Righting moment} = (\overline{GM'} \sin\theta) \times \text{weight of vessel plus cargo}, \qquad (4\text{-}5)$$

Where $\overline{GM'}$ denotes the distance along y-y by which G is below M'. The minimum distance, $\overline{GM}$, is known as the metacentric height for vessel (and cargo) when at a given displacement.

Several interesting deductions from Eq. 4-4b or 4c are as follows:

(a) With a selected maximum draft (i.e., total depth of submerged underbody) the displaced volume is approximately proportional to the beam, and BM is thus in the same degree proportional to the square of the adopted beam. Or, for a vessel of given beam, BM is inversely proportional to its total weight and displacement (and approximate draft). For commercial vessels when unloaded BM is frequently in the order of 1 1/2 feet per foot of beam, but 1/2 or less when fully loaded; it is 0.32 in the barge of Fig. 4-4.

(b) A location of the center of gravity of vessel and cargo such that GM is only about 4 or 5 percent of the beam is frequently regarded as sufficient and also desirable, as the correspondingly moderate righting moments give the vessel a longer and "easier" roll when in rougher seas. But considerably greated proportions are encountered.

(c) In apparent contradiction to the foregoing, semi-stability will still exist if <u>G</u> is slightly above <u>M</u>, but becomes coincident with the <u>M'</u> corresponding to a permissible angle of inclination. The vessel will persistently "list" at that angle, as upsetting moments operate at less angles but positive although reduced righting develop at greater angles.

Graphs showing the variation of BM' with θ, for an actual vessel and for a range of loadings and corresponding displacements, may be provided by the naval architect and be a highly useful accessory. Their form for the actual vessel will be similar to the curve of Fig. 4-5, except at inclinations sufficient that the lower edge of a deck becomes submerged or the bottom emerges, after which M' begins to recede.

Considerations paralleling the foregoing relate also to the longitudinal stability of a vessel, with respect to the longitudinal locations of B and G.

4-5. <u>Influences of Cargo Shift</u>. The analyses of the preceeding article provide a background for the consideration also of situations where the inclination of a vessel, by action of wind or wave, may have caused transverse movement of solid or liquid cargo, and consequent shift of G. Although the buoyancy is not thereby reduced, for solid cargo it changes the posture maintained by the vessel, and affects adversely its stability. The situations encountered may be quite varied, but brief attention is given to several simpler ones.

(a) <u>Shifting of Solid Cargo</u>. By reference to Fig. 4-6 and the writing of moments about the original location of G, it is seen that a transverse shift in amount x of any portion (of weight W') of a solid cargo produces for the vessel-plus-cargo a transverse migration of its center of gravity through a distance $x(W'/\sum W)$, or GG', to a new position at G'. A consequent, and maintained, "list" to inclination angle β will necessarily result such that the lines of action of the gravitational force through G' and the buoyant force through B' again coincide, both intersecting axis y-y at a prometacenter M'. But as

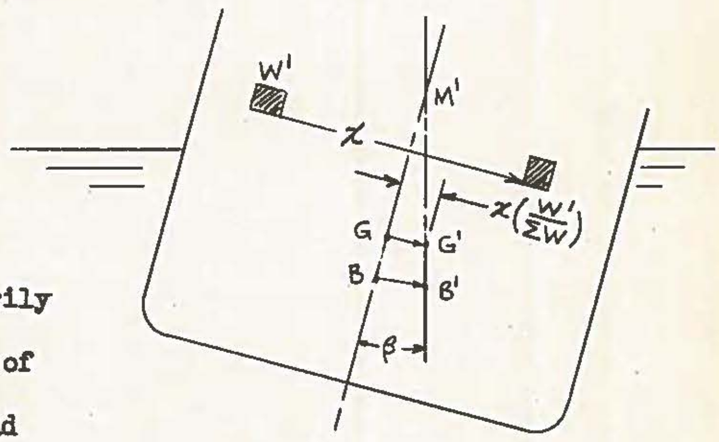

Figure 4-6

$$GG' \left[\text{ or } x(W'/\Sigma W)\right] = GM' \tan \beta$$, wherealso $GM' = BM' - BG$

and thus

$$BM' - BG = x(W'/\Sigma W) \cot \beta \quad . \qquad (4\text{-}5)$$

The righting arm introduced on any further inclination of the vessel is seen to be less than that obtainable prior to the cargo shift in the amount $(x \cos \beta_2)(W'/\Sigma W)$, where β_2 is the new inclination angle, with associated reduction of the restoring moment so introduced.

This relation will serve several purposes. One is that of ascertaining the location of G for a vessel of known characteristics by the <u>inclining experiment</u>, and is discussed in the following article. If G is established for a vessel having known characteristics as regards the weight of cargo, displacement and consequent position of B, and as the outline of the hull at the water-line determines the manner of variation of BM' with inclination, the relation enables determination of $x(W'/\Sigma W)$ if β is measured. Rectification of the list may be possible through opposite but corresponding transfer of other cargo. Otherwise it might be obtained through the admission of sea water in suitable amount into a compartment located opposite to the shift of cargo, but with reduction of buoyancy and modification of the locations of B, M and G.

On <u>lifting</u> an item of cargo through distance y, and its transverse portage through distance x, as by a crane aboard the vessel, the position of G for vessel-plus-cargo rises through distance $y(W'/\Sigma W)$ parallel to axis y-y and again shifts transversely in amount $x(W'/\Sigma W)$. By analysis paralleling the above it will be seen that the equilibrial angle of inclination will so exceed that <u>without the lifting</u> that

$$BM' = BG + (x/\tan\beta + y)(W'/\Sigma W), \qquad (4\text{-}5a)$$

with any greater inclination introducing a correspondingly reduced righting arm.

(b) <u>Flow of Liquid Cargo.</u> If liquid cargo is carried in a parallel-sided but incompletely filled compartment of a vessel (such as represented in Fig. 4-7), by analysis paralleling that of the footnote of art. 4-4 it will be seen that on inclination of the vessel the center of gravity of the contained liquid will move normal to axis y-y and (upwardly) parallel to it in the amounts, respectively, normal;

$$\left[(b')^3/12\,A'\right]\tan\beta$$

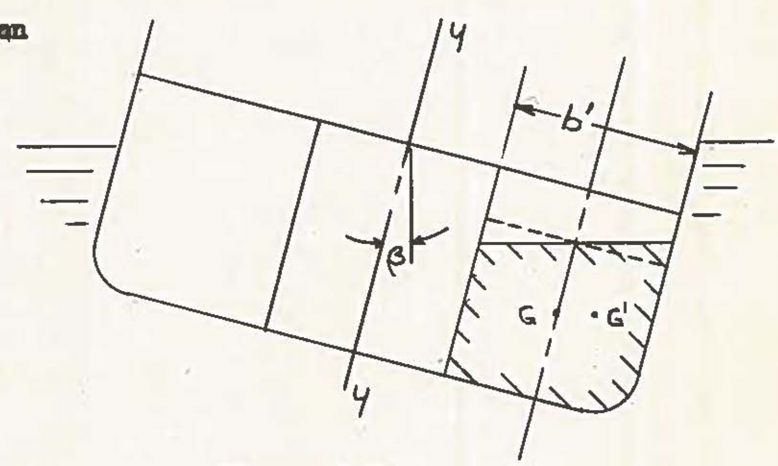

Figure 4-7

and parallel; $\left[(b')^3/24\,A'\right]\tan^2\beta$

Here b' is the breadth of the compartment and A' the transverse area of the contained mass of liquid. Except under unusual circumstances as regards the density of the fluid and the breadth and number of compartments in which the shift of gravity-center may occur, it will not act to produce a definite or constant list. However, again it operates to reduce the righting arm and moment if inclination is otherwise incurred, as by the roll of a vessel in passage through heavy seas. Also its adverse influence may be greatly aggravated by the inertial affects of the moving material if its velocity of movement is not restrained. But note that the relations are valid only if inclination neither exposes the bottom of the compartment nor causes the liquid to contact its cover. These quantities correspond respectively to the distances x and y of Eq. 4-5a, but with W' now becoming γ'A'L, where γ' is the specific weight of the liquid and L is the length of the compartment.

Significant observations in these connections are

(1) that the above effects are obviated if the compartment may be maintained full; and

(2) that, as for a compartment of given depth A' is proportional to b', the shift of the centroid of the centroid of the contained liquid is proportional to the square of the breadth. Thus there is considerable premium on keeping that as moderate as practicable.

4-6. Inclining Experiment . The naval architect may readily determine from his plans the location of B, from information on the shape and volume of the submerged portion of a vessel at any loading, and also the total weight of vessel-plus-cargo (ΣW) which will displace that volume. He may also determine values of BM, and of BM' at various inclinations, from his information on the outline of the vessel at the water-line when under given loading, and through facilities such as those of Eq. 4-4.

However, calculation of the vertical location of G and thus of the magnitude of $\overline{GM}$ for a vessel is most complex, and may be of uncertain reliability, due to the necessity of accounting for the weight and position of every item of structure and of contents. Experimental verification is thus very desirable, and is made by an inclining experiment when the vessel is afloat in undisturbed water.

The procedure is a transverse but purposeful shifting of a known weight (W') through a measured distance (x) across a deck, and precise measurement (by as long a pendulum as is practicable) of the accompanying change in the angular orientation (or β) of the vessel. The weight may be so limited as not to modify ΣW appreciably, nor to cause inclinations from the vertical exceeding about 5°, the evidence of fig. 4-5 indicating that BM' will then differ negligibly from BM. With these data, and by

Eq. 4-5,

$$BG = BM' - x(W'/\Sigma W)/\tan \beta$$

$$\text{or} = BM - x(W'/\Sigma W)/\tan \beta \; ; \qquad (4\text{-}6)$$

or in terms of the metacentric height,

$$GM = BM - BG, = \frac{x W'}{\Sigma W \tan \beta} \qquad (4\text{-}6a)$$

4-6. Period of Roll. As any manner of imposing departure from the equilibrial posture of a stable vessel sets up a restoring or righting moment, if it is permitted to do so the inertia of the vessel will cause it to oscillate about that posture. The time for a complete oscillation is called the period of roll (P).

If such oscillation may be regarded as approaching a purely rotative motion, the period and/or the angular acceleration per unit mass will be a function of the torque inciting the motion and the mass-moment of inertia (I_m) of the vessel-plus-cargo.

That is,

$$f(P, I_m, q), \text{ or } f\left[P, I_m, (W\,GM\,\beta)\right], = \text{constant},$$

where P = period of roll, having the absolute dimension of time (T);

I_m = mass-moment of inertia, of dimension ML^2; and

q = torque (i. e., righting moment) producing the motion, equalling $W\,GM\,\beta$ and having the dimension ML^2T^{-2}.

Recall that the angle (β) is dimensionless.

Restating this relation as a product of powers of the variables, but expressing these in terms of their absolute dimensions,*

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* - For exposition of this and the following technique the reader must unfortunately be referred foreward to art. 6-6.

$$f\left[T \quad (ML^2)^a \quad (ML^2T^{-2})\right] = M^0L^0T^0$$

But observe that for the requisite dimensional homogeneity of this relation, a = - 1/2 and b = 1/2. Thus, re-expressing the initial relation in terms of relevant variables,

$$P = \text{constant} \times \left[I_m/(W \quad GM\beta)\right]^{1/2} \quad \text{, or} \qquad (4\text{-}7)$$
$$P^2\beta = C \times \left[I_m/(W \quad GM)\right]$$

Equation 4-7 may be recognized as a form which might be taken by the 2nd law of motion if adapted to conditions involving oscillatory acceleration.

Although involving simplifying approximations, experience has well justified consideration of evidences offered by this relation. Significant ones are that

(1) to obtain, for a vessel of given weight when loaded, a longer and generally more comfortable period of roll (but also one of greater amplitude, or greater $P^2\beta$), the architect will wish to provide an underwater form and mass distribution such as to give a lesser metacentric height (GM); but also that

(2) a dangerous lessening of GM and thus of remaining stability, in consequence of casualties experienced by a vessel, will be indicated by a marked increase of its period and amplitude of roll.

IV-6. Problems.

1. A cylindrical metal float 4 inches in diameter and weighing 4 pounds is used to indicate the water level in a tank. How deep will the float sink, with axis vertical, and what change in submergence will need to occur in order to exert a force of 0.2 pounds on some mechanism attached to the float? ans. (8.8"; 0.44")

2. For a ship of 10,000 tons total weight (2240 lb/ton) what volume of water will it displace when floating in fresh water, and what when in sea water, at 62.4 and 64 lb/cu. ft. densities respectively?

3. Ice has a density of 57.4 lb/cu.ft. What percentage of the volume of a mass of ice is submerged when floating in sea water? Ans. (89.7%)

4. An inverted steel tumbler floats in mercury. The tumbler is cylin-

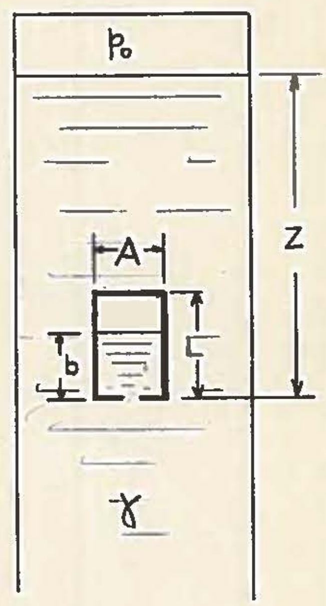
drical, with one end closed; its length is 4.5", its radius 1" and its weight 3 pounds. It is observed that the mercury rises 0.27" in the inside of the tumbler.

a) How much of the tumbler protrudes from the mercury?

b) What is the air pressure inside the tumbler?

Ans. (2.28", 15.66 psia)

5. When the pressure po, in the pictured "Cartesian Diver" is increased, the amount of water in the small "diver" increases and it sinks. When po is reduced sufficiently, the diver returns to the surface. For a given pressure po, weight of empty diver W, and size A x L, specific weight of fluid, γ the diver has an equilibrium position at same depth Z. Find this depth if:



$A = 1\ ft^2 \qquad W = 20\ lbs.$

$L = 1\ ft. \qquad \gamma = 62.4\ \frac{lbs}{ft^3}$

p_o = 2 atmospheres

Consider that the mass and temperature of the entrapped air are constant and that, therefore, the product of the absolute air pressure and its volume will be constant, commencing from a full container of atmospheric pressure air. Ans. (38.7 ft.)

6. Is the "diver" of problem 5 rotationally stable? Is it stable in the vertical direction?

7. A 10' wooden plank of specific gravity .5 has an 8" x 12" cross section. Find the righting moment when it floats such that its cross sectional diagonal lies in the plane of the water surface.

8. The outside dimensions of a barge are 90 feet length, 20 feet beam and 7 feet depth. Its weight is 60.7 tons of 2240 pounds, with its center of gravity located 1.50 foot above the bottom. Determine the following items when in sea water the barge is in the stated conditions of loading and inclination.

(a) Depth of submergence of bottom when light and when symmetrically loaded in the amount of 150 tons (2240 lb), and the height of B above the bottom (KB) in both situations. (1.18 and 4.10 ft; .59 and 2.05)

(b) The metacentric radius BM and the corresponding height of M above the bottom (KM) for both draft conditions, and the locations of the prometacenters (i.e., KM') when loaded if the barge is inclined at 6, 10, and 15° and no shifting lf the load results. (28.25; 8.13)

(c) The metacentric height (GM)when light, its ratio to the beam and the righting moment at an inclination of 6°. (27.34; 388,000 lb-ft)

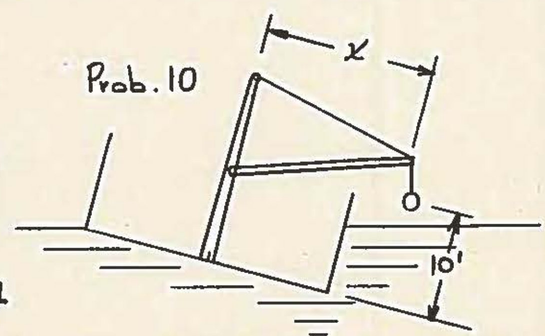
(d) The distance of G from the bottom (KG) when loaded and the distance of the center of gravity of the cargo from the bottom is (1) 3.0 ft and (2) 9.5 ft., and the corresponding values of the metacentric height. (2.56 and 7.19; 7.62 and 2.99)

(e) The righting moment at both conditions of loading for inclinations of 6°, 10° and 15°. (Note that when loaded the edge of the barge bottom would be at the water surface at the inclination of 16.2°

(964,000 and 378,000 lb-ft at 15)

9. For the barge of problem 8 when loaded in the several manners as specified, to what distance from the axis might the load be shifted if the resultant inclination is not to exceed 15°? (2.87 and 1.125 ft)

10. On the barge of problem 8, but provided with 15 tons of ballast so located that the center of gravity remains at 1.5 ft. above the bottom, it is proposed to rig a hoist capable of lifting 15 tons to a height such that its distance from the plane of the bottom is 10 ft. To what distance outboard from axis y-y might this weight be swung if the resulting inclination shall not exceed 10°, at which inclination the edge of the bottom of the barge will be at the water surface under the stated loading? Would the provision of additional ballast increase or decrease this distance? (18.3 ft)



11. A homogeneous block of specific gravity 0.8 has dimensions: height a, width b, and unit depth.

a) Find the ratio of a to b for that block which will just become unstable (in the a b plane) at an inclination of 30 degrees.

b) Find the same ratio for a block of specific gravity 0.9

Ans. (1.1, 1.56)

12. The barge of problem 8 is carrying 35,000 gallons of oil (sp.gr. = 0.9) equally distributed between six (6) transverse tanks of sizes 18 ft x 5 ft. The bottom of the tanks is 2 ft. above the bottom of the barge. What would be the righting moment at an angle of inclination of 15° if (a) the tank covers are well above the high side of the oil, (b) the covers are well above the oil but longitudinal bulkhead are provided in the tanks separating each into two equal compartments, and (c) the tank covers are so located that all tanks are filled to the covers by the above amount of oil?

13. In an inclining experiment on the barge of problem 8, with the load in the lower position, the movement of a supplementary 2-ton mass from the center line a transverse distance of 9 ft caused an inclination of 0° 38'. The center of gravity of the mass was 8 ft above the bottom. Check the value of GM as computed in the problem.

PART III

FLOW ENERGETICS

Chapter 5. Continuity, Energy and Momentum Relations

5-1. Foreward. The preceding chapters have dealt with hydrostatics; that is, the consequences of the earth's gravitational force-field on a fluid mass, but when unaccompanied by relative motions between its particles. While of definite practical significance, a much broader field of engineering endeavor is concerned with the behaviors when relative motions instead exist. Various forces, including gravitational, operate to produce specific fluid motions, or conversely, fluid motions are modified for the purpose of generating useful forces and associated power. In some instances the analyses of such phenomena are better made by focusing attention on individual fluid particles. This "microscopic" approach is used in later portions of this material, but it may often involve impracticably complex analyses or otherwise require simplifying but questionable assumptions. A less rigorous but most useful "microscopic" approach makes certain justifiable engineering approximations in examining large groups of particles, rather than individual ones. This approach is taken initially since it lends itself more readily to clear physical interpretation of fluid phenomena, but it should be added that the two approaches are not at all incompatible. The solution of many engineering problems may involve the results of both types of analyses. They are complementary rather than exclusive.

Principles providing the basis for the microscopic analyses are the familiar ones known as the Conservation of Energy and of Mass*, and one referred to as the Impulse or Momentum Principle. Adaptations of these in the form of energy, continuity and momentum equations are the subject matter of this and several subsequent chapters.

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* - Disclosures of modern physics have established that a)mass is in fact a manifestation of energy and capable of transformation to energy by nuclear reactions such as fusion or fission, and that b) the mass of a particle is likewise dependent upon its velocity. However, engineering fluid mechanics is not concerned with the former, nor are the velocities of encountered sufficiently high that the latter is of significance. Hence here the conservation principles may suitably be regarded as individually valid.

5-2. Distinguishing Characterizations, Fluid Streams. Throughout subsequent material it will be necessary to employ various terms which are conventionally used to express certain essential and distinguishing features of fluid streams. Some of these features are more or less realistic, in the sense of being closely attainable even under actual flow conditions, but some relate to idealized conditions and to concepts of great utility although in actuality capable only of approach. They will be given introductory attention at this point, but possibly with some sacrifice of rigor and completeness permitted for purposes of simplicity.

(a) Steady, versus Unsteady, Flow. The term steady flow signifies that neither the mass-rate of flow at a given section or throughout a stream nor the physical state of the fluid changes from moment to moment, or is thus said to be invariant with time. It is in contrast to conditions in which that rate is instead varying and the flow is thus said to be unsteady. If interest is restricted to the flow of liquids, of effectively constant density, constancy of the volume-rate of flow becomes a valid alternative index of steadiness.

To illustrate, steady flow may be regarded as attained in a situation such as the passage, through a centrifugal pump which is operating at constant speed and constant pressure (and temperature) in its intake and discharge channels. But it becomes unsteady if any of these are instead in process of changing.

But note in these connections that the question of steadiness or unsteadiness may depend on the viewpoint of the observer of the flow. This will become a matter of some subsequent concern. For example, to an observer mounted on some stationary obstruction about which an open stream is passing steadily the character of its flow may appear to remain the same from moment to moment, but to an observer mounted on a float which advances with the stream and follows its path, the character of the flow as he approaches the obstacle would appear to differ quite definitely from that when he is subsequently passing it. The same flow conditions would appear to him to be unsteady.

(b) Uniform, versus Nonuniform, Flow. By uniform flow is meant that occurring in a stream for which the velocity does not change either in magnitude or direction as the stream proceeds through space, or is thus said to be invariant with respect to space. If such changes do occur, as in passage through a nozzle or about a bend, the flow is said to be nonuniform. Attainment of uniform flow throughout an actual stream is not too common, but the term is frequently used in a looser sense as descriptive, for example, of the flow of a liquid through a straight pipe of constant cross-sectional area.

(c) One, Two and Three-Dimensional Flow. The term one-dimensional flow denotes one in which, at a given cross-section of a stream, all parts may be regarded as proceeding in a single direction, with the magnitude and direction of its velocity thus representable by a single vector. The flow may be steady or unsteady but is necessarily uniform, and might be thought of as ideally proceeding through a straight channel having parallel sides.

Two-dimensional flow is simply one in which the stream velocity needs be regarded as having components only in two directions, such as the x and y directions if employing cartesian coordinates or the radial and tangential directions to polar coordinates. The flow may be steady or unsteady but is evidently nonuniform, and might be thought of as ideally proceeding through a channel having two parallel sides but, if of rectangular cross-section, with the other two sides converging, diverging or curving.

Three-dimensional flow is similarly one in which the stream velocity has components in the three (x, y and z) directions, and might be thought of as proceeding through a channel for which none of the sides are parallel.

In more formal literature the velocity components in the x, y or z directions are conventionally distinguished by the respective symbols u, v and w. In this text, and for economy in the expenditure of letter symbols, these may frequently be distinguished instead by subscript notations such as u_x, u_y or u_z; also using the symbol u without subscript for velocity designation in those frequent circumstances in which directional considerations need not be emphasized. In current engineering literature the symbols u or V are frequently employed for the same purpose.

With few exceptions subsequent considerations will be limited to one-or two-dimensional and steady flow conditions. Although three-dimensional ones are in fact more realistically representative of actual situations, their treatment normally requires highly complex kinematic and dynamic analyses, while analysis of two-dimensional flow is more manageable and is adequate for many applications.

(d) Laminar versus Turbulent Flow. Laminar flow, known also as viscous flow, is one in which the velocity distribution across a stream is established solely by the restraint to relative motion between adjoining layers of fluid imposed by the fluid viscosity (μ, art. 1-8) It is characterized by no perceptible passage of the fluid between the adjacent layers, although accompanied by that interchange of molecules, in transverse travels of infinitesimal length, to which the viscosity property is in part attributable. Laminar flow often exists when a stream is passing through a confining channel at very moderate rates. At higher flow rates the main part of the stream is no longer laminar, but laminar flow persists in a thin layer which adjoins any surface past which the fluid is passing and constitutes the major portion of a boundary layer.

In streams passing along a surface at higher rates a secondary influence of viscosity develops which causes the fluid particles to depart from the laminar flow configuration, and to induce what may be thought of as a "rolling-up" or "snow-balling" character of motions, described technically as vortices, in the flow field beyond the boundary layer. In this situation the transverse travels at molecular levels are supplemented by more-or-less organized transverse components of motion, superimposed on the major motion of the stream in its directions of advance. Such flow is known as turbulent, and is typically encountered in engineering installations.

In these connections a third but idealized type of stream flow may at least be noted, although more specific consideration of it does not appear until later. For reasons which will then appear, it is referred to as irrotational or potential flow* and at this point will be defined simply as

* - or by the thermodynamicist, as mechanically reversible.

that character of flow assumed by a fluid which is idealized in the feature of having zero viscosity. Although the flow is never actually attained, the concept is a highly useful one. Irrotational flow may be closely approached, or in situations where influences other than viscosity (e.g. inertial or gravitational) are predominant in establishing the pattern of a real-fluid flow.

A stream of finite magnitude may conveniently be regarded as the aggregate of a multiplicity of stream tubes across the boundaries of which no flow exists.

(e) The Stream Tube and Stream Line. A stream of finite magnitude may conveniently be regarded as the aggregate of a multiplicity of (possibly infinitesimal) stream tubes, across the boundaries of which no flow exists. A related concept having exceptional utility, both for informative picturization of the "pattern" of a steadily flowing stream and for analytical purposes, is that of the stream line.

In relation to the stream tube, stream lines may be regarded as ones delineating the boundaries of the tube. As such they serve to represent the path followed by any fluid particle at the boundary, with a tangent drawn to the line (if curved) at any point so serving also to indicate the local direction of motion of the particle. But in two-dimensional flow, and in some cases for three-dimensional, the relative separation of neighboring lines will serve further to indicate the (mean) velocity in the tubes they bound, if those tubes are regarded as ones through which the volume-rates of flow are equal.

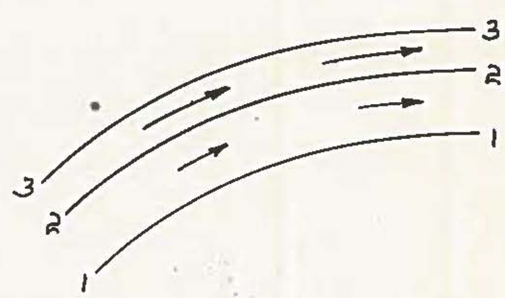

Figure 5-1

Figure 5-1 illustrates the greater separation between stream lines 1 and 2 denoting a less mean velocity than in the adjoining tube bounded by the more closely spaced lines 2 and 3. By this technique the distribution, the direction and the magnitude of the velocities in a stream having steady but non-uniform flow are jointly representable. The jumble of vectors which would otherwise be required if attempting to indicate these essential characteristics of the stream is so escaped.

5-3. The Continuity Equation. As a working adaptation of the principle of the conservation of matter in relation to flow phenomena, the continuity equation is simply one equating the mass-rate of entry to a region or device to the rate of departure plus the rate of change of the quantity of material remaining within the confines of the region. For present purposes it is however sufficient to limit attention to instances of steady flow, in which rates of entry and departure are equal and no accumulation or diminishments of the material within the region is thus relevant; i.e., for steady flow,

$$\dot{m}_{in} = \dot{m}_{out} \ , \text{or} \quad \dot{m}_1 = \dot{m} = \dot{m}_2$$

where $\dot{m}$ (read "m dot") denotes the mass-rate of fluid passage via an entry point 1, any intervening point, and departure point 2. Or in terms of the volume rate ($\dot{v}$),

$$\dot{V}_1 \rho_1 = \dot{V}\rho = \dot{V}_2 \rho_2 \ ;$$

and for liquids of constant density, $\dot{V}_1 = \dot{V}_2 = \dot{V}_3$

An advantageous modification of these simple relations for one dimensional flow is provided if, denoting by <u>A</u> the transverse cross-sectional area of a stream, one recognizes the quotient $\dot{m}/\rho A$ as a suitable evaluation of a <u>mean</u> rate of advance, or mean velocity $\bar{u}$, with which the fluid proceeds. Doing so,

$$\dot{m} = \rho_1 A_1 \bar{u}_1 = \rho A \bar{u} = \rho_2 A_2 \bar{u}_2 = \text{constant} \qquad (5\text{-}1)$$

or in terms of the volume rate for a negligibly compressible liquid,

$$\dot{V} = A_1 \bar{u}_1 = A \bar{u} = A_2 \bar{u}_2 = \text{constant} \qquad (5\text{-}1a)$$

The employment of a mean velocity as so interpreted is customary and most advantageous for engineering purposes, as considerations of more exacting nature are not of immediate concern, but it is well to recognize that, aside from the restriction to steady-flow conditions, no consideration has been given to situations in which velocity components in a second or third direction require attention. For more comprehensive analyses it becomes necessary to provide more adequate but also more elaborate forms of the continuity equation.

5-4. <u>Energy Classifications</u>. For enabling requisite energy accounting when fluid flow is involved it is essential first to have adequate awareness of the several sorts of energy manifestations with which we shall be concerned. These may be recognized to advantage as being of two distinctive characters, the classification being based on whether the energy may be <u>attributed in definite amount to a unit mass of the fluid</u> when in a given mechanical or internal status, or whether the energy is instead <u>in transition between the mass and its environs</u> (but with consequent influence on the status of a departing fluid).

The energies to be considered in these two categories are

(A) <u>Energies allocatable in definite amount to unit mass when in a given status</u> -

(a) Potential energy of position, or geopotential energy;

(b) Kinetic energy, associated with orderly motion of the stream;

(c) "Internal" energies; and

(d) Flow-work energy.

(B) <u>Energies in transition to or from the environs of the stream</u> -

(e) Shaft work; and

(f) Heat.

Considerations relating to these individually are indicated below.

5-5. <u>Potential Energy of Position, or Geopotential Energy.</u> The concept of a potential energy attributable to a mass by virtue of a superior elevation, normally relative to the earth's surface and so described as <u>geopotential energy</u>, follows from either (a) its known ability to produce quite sensible effects if the gravitational attraction is permitted to operate in retracting it to a lower elevation, or (b) awareness of the effort that has been initially required in effecting the elevation. Evaluation of the relative amount of energy that may so be ascribed to the elevated mass, although more exactly to the gravitational system of which the mass is a component, is more readily accomplished by consideration of the second viewpoint. That is

$$\left.\begin{array}{l}\text{Energy required to}\\ \text{elevate, or change}\\ \text{in geopotential}\\ \text{energy}\end{array}\right\} = \begin{array}{l}\text{Vertical force effecting the}\\ \text{elevation times vertical}\\ \text{distance through which}\\ \text{elevation proceeds}\end{array}$$

$$= \begin{array}{l}\text{weight of fluid elevated}\\ \text{times increase in elevation}\end{array}$$

$$= (m\ g)\ (z - z_o),$$

where g is the local acceleration due to gravity but also the associated local weight of unit mass of the fluid, and $z - z_o$ denotes the increase in elevation. It frequently becomes more convenient to have the relation in terms of the change in geopotential <u>per unit mass</u> or

$$\begin{array}{l}\text{Change in geopotential} \\ \text{energy per unit mass}\end{array} = g\,(z - z_0), \qquad (5\text{-}2)$$

where the units are consistent ones, such as lbf/slug and feet, dynes/gram and cm, or newton/kg and meters. Or in the inconsistent but common engineering units of lbf/lbm and feet,

$$\begin{array}{l}\text{Change in geopotential energy,} \\ \text{ft-lbf per lbm}\end{array} = (g/32.17)(z - z_o) \qquad (5\text{-}2a)$$

Recall that the energy per unit mass has the basic dimension L^2T^{-2}.

5-6. <u>Kinetic Energy.</u> Mechanical kinetic energy may similarly be credited to a mass of fluid whose particles are in such unidirectional and coordinated motion that, as with a "jet" of water, the stream is able to produce mechanical effects. Although the concept that kinetic energy is assignable to a moving stream is not inherently dependent on consideration of the means whereby the energy may have been acquired, its evaluation is more readily provided by regarding the motion as the result of the operation of an accelerating force on the mass from rest to an existing velocity. The energy so supplied to the moving mass equals $\int_0^L F\,dL$.

Expressing the force in terms of the mass accelerated and its rate of acceleration, or $F = k\,m\,a$, and associating the concepts of distance, velocity and acceleration in the defining relations $u = dL/dt$ and $a = du/dt$, whereby $dL = u\,dt = (u\,du)/a$,

$$\text{Kinetic energy} = \int_0^u k\,m\,u\,du = k\,m\,u^2/2;$$

and

$$\text{Kinetic energy per unit mass} = k\,u^2/2, \qquad (5\text{-}3)$$

in which it will be recalled that k = unity when employing consistent units, but 1/32.17 when the inconsistent pound force and pound mass units are used.

The kinetic energy per unit unit mass of a fluid stream having a given velocity is thus a function only of that velocity. But if, when employing this expression in relation to an entire stream, the velocity is taken as the <u>mean</u> velocity of Eq. 5-1 or 1a (i.e., $\bar{u} = \dot{m}/\rho A$), it is wholly valid only if all portions of the stream are moving at the same velocity. Sketch (a) of Fig. 5-1 indicates such a condition, and would be representative only for the uniform flow of an <u>ideal</u> (non-viscous) fluid. For the infrequent but more realistic situation of the laminar flow of a real fluid, as represented in sketch (b) for such flow through a circular pipe, an

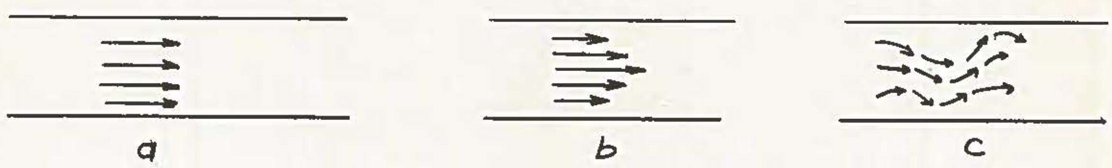

Fig. 5-2

interpretation of the mean kinetic energy as $(\dot{m}/\rho A)^2/2$ would precentage wise to appreciably in error*. But in such flow (*see footnote, page .) the kinetic energy is normally so minor a portion of the total energies that in the aggregate the error may well be negligible.

With the more rapidly flowing, turbulent streams that are typical in engineering processes, the transverse distribution of the axial components of velocity may be somewhat as indicated in sketch (c) of the figure. Computation of the mean kinetic per unit mass as $\frac{\bar{u}^2}{2}$ in this case involves but minor error. In practive these effects are commonly compensated by the use of experimentally determined corrective coefficients,

The kinetic energy attributable to the transverse and more or less disorganized motion-components of the turbulent stream rapidly degenerate into the wholly haphazard molecular activity which is to be regarded as a major portion of the <u>internal energy</u> of a fluid.

5-7. <u>Internal Energy</u>. It has just been implied, was further noted when considering the concept of the dynamic viscosity of fluids, and is amply verifiable otherwise, that at temperatures above the absolute zero the molecules comprising any material are in a state of incessant but disordered motion. The energy attributable to such molecular activity is suitably regarded as kinetic in character, and is one of a number of energy manifestations which may in fact be stored <u>within</u> a fluid and so described as <u>internal energy</u>. In the following this will be represented, in its amount per unit mass of the material, by the symbol $\underline{e_i}$.

In this connection, the mechanical kinetic energy of the last article is in a sense to be regarded as a measure of those components of otherwise haphazard molecular motions which, through suitable directive actions, have been caused to be uni-directionally organized, and thereby of mechanically useful character. But also, any flow conditions which operate to degenerate the ordered to disordered molecular motions are properly to be regarded as acting to increase the internal energy.

Increase of that portion of the internal energy of a fluid is evidenced by increase in its temperature. For example, for water at atmospheric temperature levels, the temperature increase is about one degree Fahrenheit for 778 ft lbf of energy increase per pound mass of water, or 25,000 ft lbf per slug. In the thermodynamic studies relating to the flow of expansible fluids direct attention needs be given to the internal energy store, since with such fluids it may in part be converted to mechanically useful energies; but in studies concerned with incompressible liquids, such specific attention is generally unnecessary. However, an awareness of it, as a repository of energy of less evident character, is essential in connection with any proposed energy accounting.

5-8. Flow-Work Energy. The relative amounts of geopotential, kinetic and internal energies stored by unit mass of a flowing fluid are easy to visualize physically and to measure quantitatively. But in flow phenomena there invariably is yet another pertinent energy item which must be considered in any energy accounting. Although not to be regarded as stored in the same sense as the other energy items, it is still assignable in definite amount per unit mass. It arises when a fluid stream is made to advance against an opposing pressure force* and is known as flow-work energy. Its physical reality would become quite evident if, for example, in the manual operation of a pump, the pressure in the region into which the fluid were being delivered were suddenly increased, but the rate of flow of fluid was maintained constant.

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* - A clear distinction must be made between this and such internal energy as will previously have been stored when compressing the fluid to the pressure required for effecting the flow.

Evaluation of the flow-work energy per unit mass of fluid is provided by survey of the situation represented in figure.

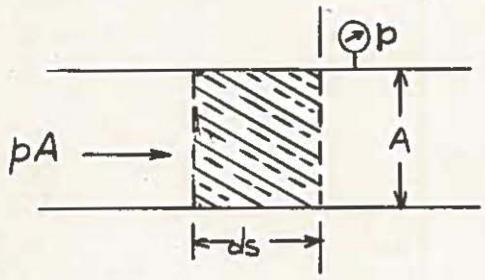


Fig. 5-3

5-3. There a parcel of fluid of area A (transverse to its direction of advance), of length ds and corresponding volume A ds, and of density and corresponding mass A ds, is advancing horizontally and without accelation against a restraining pressure-force equalling p A. For advance of the mass past a transverse section in plane x-x a force of this magnitude must operate through distance ds. The energy so required, as flow-work, is p A ds; or the flow-work per unit mass of fluid so advanced becomes

$$\text{Flow-work per unit mass} = (p\ A\ ds)/(\rho\ A\ ds)$$

$$= p/\rho \quad , \text{ or } p\ v. \qquad (5-4)$$

Consistent units for evaluation are ones such as ft-lbf/slug, lbf/sq. ft. and slugs/cu. ft; ergs/gram, dynes/sq. cm. and grams/ cu. cm; or newton-meters/kg, newtons/sq.m. and kg/cu.m.

It is noteworthy in this connection that, if using the pound-force and pound-mass units, for which (by Eq. 2-1) $dp = -\rho(g/32.17)\ dz$ in a static column of fluid and as numerically g/32.17 unity at the earth's surface, the ratio p/ρ in such units also equals numerically the linear height of a column of fluid of density ρ which would be supported by a pressure p at its base. A resulting practice has been to express the flow-work energy as a "pressure head", evaluated in "feet" of the relevant fluid. Although this terminology has some convenience, and is a measure actually of the work per unit (local) weight per unit mass of fluid, it is in general preferred to retain the flow-work and energy significance of the p/ρ item.

Quite regularly the geopotential and flow-work energies at a given location in a liquid stream are jointly significant and their sum $(gz + p/\rho)$ becomes a convenient single item when writing an energy account. It will here be represented by the single symbol e_{zp} when that is advantageous.*

* - In thermodynamic analyses, and for like reasons of convenience, the sum $e_i + p/\rho$ is given the single designation of enthalpy and denoted by the symbol h.

It has also been known as the "piezometric head", as the distance $z + p/\gamma$ of Fig. 5-4 is effecting the lbf and lbm units of force and mass.

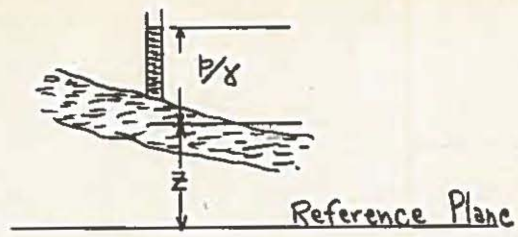


Figure 5-4

5-9. Shaft-work, and Heat. In his practice the engineer is often concerned with situations in which flow is maintained through the agency of work input via the shaft of a pump, or in which energy is extracted from a stream in the form of work delivered through the shaft of a turbine. In such circumstances an energy accounting will obviously require inclusion of such work terms. In the following they will be referred to by the self-descriptive term shaft-work and represented, per unit mass of the fluid, by a symbol such as ${}_1w_2$. The leading and following subscripts signify that the energy entry or departure occurs while the fluid stream is in passage between a first and a second section, but may not be assigned to unit mass at either individually.

In thermodynamic studies concerned with heat-power and associated equipment it is further necessary to provide for those situations in which energy passes to or from a fluid stream by reason of a deficit or an excess of its temperature relative to that of its environs. Such energy is described as heat and denoted, per unit mass of fluid, by symbol such as ${}_1q_2$.* Although the chance of such energy transition needs to be recognized in fluid-mechanics studies, and may occasionally become significant, in the majority of instances its relative amount is negligible, and the flow may be said to be adiabatic.

5-10. Steady-flow Energy Equation. Having noted in preceding articles the various energy manifestations which are pertinent in flow phenomena, and the manners of their evaluation, we are in a position to incorporate them into an equation based on the conservation-of-energy principle. It is exceptionally

* - Although the word heat is at times employed to designate both such transitional energy and the internally-stored molecular energy of art. 5-7, current engineering literature avoids the confusions arising from such dual usage and restricts its significance to the above.

well adapted to the indicating of their mutual influences in connection with the steady flow of a fluid stream through representative devices. Admittedly it is completely applicable in principle only to the ideal situation of steady, uniform and irrotational flow. But, because of its great utility, it is reg-employed in connection with the flow of real fluids through real devices, and associated approximations are tolerated or even welcomed, but otherwise accounted for.

For practical development of the steady-flow energy equation refer to Fig. 5-5, in which is represented diagrammtically some device through which steady flow is occurring. Flow is to and from the device via the indicated inlet and outlet channels and across transverse sections (1) and (2) in those respective channels. The device may be a pump, a turbine, a nozzle, or merely an interconnecting pipe. Data are presumably available as regards the elevations relative to some selected reference plane (z), the mean velocities ($\bar{u}$), the pressures (p) and the densities (ρ) at sections (1) and (2). If the device were a pump or turbine a necessary appurtenance would be a shaft through which energy might enter or depart, as shaft-work. Also energy might enter or depart as heat if the temperature of any portion of the device or the fluid differed materially from that of its environs.

An accounting of the energies entering or departing from the indicated <u>system</u> may be written from either of two points of view. One regards the energy quantities as associated with some unit mass which enters at (1), experiences a change of state and departs at

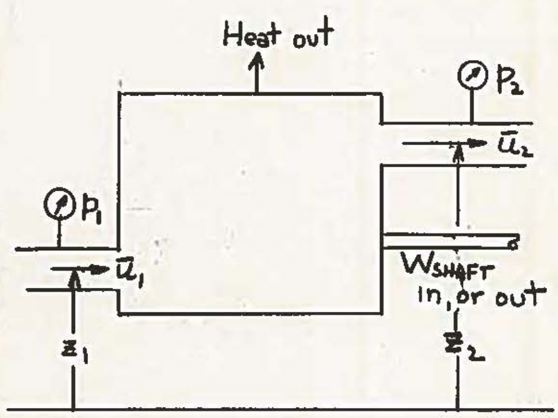


Fig. 5-5

(2). The other considers the energy quantities to be associated with the different masses which at the same instant are entering at (1) and departing at (2). From either viewpoint,

Energies entering via section (1) through the agency of entering fluid	$\pm$	Energies entering or departing as work (or heat) while enroute from section (1) to (2)	=	Energies leaving via section (2) through the agency of leaving fluid

Expressing specifically the individual energies to be accounted, and presuming the use of consistent units,

$$g(z_1 - z_0) + \frac{\bar{u}_1^2}{2} + \frac{p_1}{\rho} + e_{i,1} + {}_1w_2\,(+, \text{in}; -, \text{out}) + {}_1\text{heat}_2 = g(z_2 - z_0) + \frac{\bar{u}_2^2}{2} + \frac{p_2}{\rho} + e_{i,2} \qquad (5\text{-}5)$$

As the specification of steady flow connotes that, at an given location in the stream the state of the passing fluid remains constant, no accumulation nor diminution of the energy stored within the system may occur.

Scrutinizing the equation, it is evident that the z_0-term will disappear and might well be omitted, this indicating that for the geopotential energy accounting any convenient reference plane might be employed. When applying the relation to the flow of negligibly compressible liquids the density term may be written without subscripts. Individual accounting of the internal energy terms is not made, nor of possible energy transmissions as heat. Instead these will be accounted for by a single term, denoted as ϕ and representing the composite ($e_{i,2} - e_{i,1}$ + heat), and also described as the <u>dissipated energy</u>. Recall that increase in the internal energy in real-liquid flow was recognized above as attributable to viscous "drag" and resulting degeneration of useful energies to disordered molecular activity. In the flow of liquids any energy received as heat has normally the same fate*.

With these adjustments, and for the flow of a liquid,

$$g z_1 + \frac{\bar{u}_1^2}{2} + \frac{p_1}{\rho} + {}_1w_2\,(+, \text{in}; -, \text{out}) = g z_2 + \frac{\bar{u}_2^2}{2} + \frac{p_2}{\rho} + \phi \qquad (5\text{-}5a)$$

This relation is used so extensively in the following that full understanding of its intecedants and significance is essential.

The conclusions of Eq. 5-5a were in part formulated by Daniel Bernoulli several centuries ago (1738), but in a relation omitting the shaft-work and energy-dissipation terms and so limiting its utility to flow situations in which no energy entry or departure as work in pump or turbine is anticipated, and to idealized flow conditions. That is, the <u>Bernoulli equation</u> as frequently quoted in fluid-mechanic literature is of the form

---

* - In thermodynamic analyses involving the flow of compressible fluids, energy reception or departure as heat requires more exacting accounting but also techniques beyond the suitable scope of this material.

$$gz_1 + \frac{\bar{u}_1^2}{2} + \frac{p_1}{\rho} = gz_2 + \frac{\bar{u}_2^2}{2} + \frac{p_2}{\rho} \qquad (5-6)$$

The following illustrative example indicates the utilization of the steady-flow energy relation.

Example 5-1. A given pump has a suction line 12" in internal diameter and a discharge line of 10" diameter. It is delivering fresh water having a density of 1.940 slugs (62.4 lbm) per cu. ft. under steady operating conditions as follows:

Rate of delivery; 4,000 gal. per minute (1 cu. ft. = 7.48 gal). Pressure in suction line near pump; -3.0 psi gage. Pressure in discharge line near pump, but at a point 2 feet above that at which the suction pressure was measured; 75 psi gage.

Determine the following items -

(a) Work required to drive the pump per pound (mass) of water delivered, and that per second, and the corresponding horsepower, if the pump were an ideal one in the sense that no energy dissipation due to friction and turbulence occurs in passage through the pump (known also as the hydraulic horsepower output).

(b) Work and power required if 25% of the work actually supplied were dissipated in friction and turbulence, with consequent internal energy and temperature increase of the water in passage; or the pump was so to be described as having 75% efficiency.

(c) Approximate temperature rise of the water while enroute through pump.

Solution.

Rate of delivery = 4000/(60 x 7.48) = 8.91 cu. ft./sec.
or = 8.91 x 1.940 = 17.29 slugs or 557 lbm per sec.
Suction line, area = $(\pi/4)(12/12)$ = 0.785 sq. ft.
Suction line, velocity $(\bar{u})$ = 8.91/.785 = 11.35 ft/sec
Suction line, kinetic energy = $(11.35)^2/2$ = 64.4 ft-lbf/slug
Discharge line, area = $(\pi/4)(10/12)^2$ = 0.545 sq. ft.
Discharge line, velocity = 8.91/.545 = 16.35 ft/sec
Discharge line, kinetic energy = $(16.34)^2/2$ = 133.7 ft-lbf/slug
$(p_2 - p_1)/\rho$ = 144(75+3)/1.94 = 5790 ft-lbf/slug, or 180 ft-lbf/lbm
$g(z_2 - z_1)$ = 32.17 x 2 = 64.4 ft-lbf/slug

(a) Work input, ideal = $g(z_2 - z_1) + (u_2^2 - u_1^2)/2 + (p_2 - p_1)/\rho$
= 64.4 + (133.5-64.4) + 5790 = 5924 ft-lbf/slug
and = 5924 x 17.29 = 102,430 ft-lbf/sec
Horsepower, ideal (or "hydraulic") = 102,430/550 = 186.2 hp

(b) At ϕ = 0.25 or $W_{actual} = W_{ideal} + 0.25\ W_{actual}$,
Work input, actual, = 5924/(1-0.25) = 7898 ft-lbf/slug
Dissipated energy = 7898 - 5924 = 1974 ft-lbf/slug
Horsepower, actual, = (7898 x 17.29/550 = 248.3 hp.

(c) Temperature rise = 1974/25,000 = 0.079°F.

5-11. Application, Energy and Continuity Equations. Joint engineering utilization of the (steady-flow) and continuity equations was in fact illustrated in Example 5-1, there for ascertaining the required input to accomplish certain specific objectives. In other situations, for example, the determination of attainable flow rates, several further considerations may require attention. Such are indicated here, and a following example illustrates.

(a) Data must obviously be available of suitable and sufficient character that the number of unknowns is not such as to make their determination impossible. But it frequently occurs that, by strategic selection of the locations between which the equations are <u>initially</u> written, an apparent inadequacy of data may in fact be found to be sufficient.

(b) When evaluating the flow-work term (p/ρ) the pressure item is in principle to be interpreted as the total or <u>absolute</u> pressure, and <u>must</u> be so interpreted in flow analyses for gases whose density is variable and a direct function of the atmosphere, or the <u>gage</u> pressure. However for liquids having negligible density change with change in pressure, difference-quantities such as $(p_2 - p_1)/\rho$ equal also $[(p_{2,\,gage} + p_{atm}) - (p_{1,\,gage} + p_{atm})]/\rho = (p_{2,gage} - p_{1,gage})/\rho$. It so became quite customary in <u>hydro-mechanic</u> analyses to express p in terms of the more directly measurable gage pressure.

When following this practice negative values of p_{gage} may readily be encountered, and in fact express the amount by which the total pressure is less than atmospheric, however. The apparently negative, or less-than-zero, values of the flow-work energy which will so seem to be indicated are in principle fictatious or at least <u>nominal</u>, although expressing the total energy <u>less</u> that which has been contributed more or less directly by the atmospheric pressure in effecting the advance of the stream.

(c) Due caution is to be exercised that items relating to a particular location are not ascribed also or instead to another which chances to be quite near in space.
In a representative stream convergence such as is indicated in Fig. 5-6, occurring in the flow through an aperture of the form indicated, the pressure and velocity at section (1) may not be attributed also to section (2).

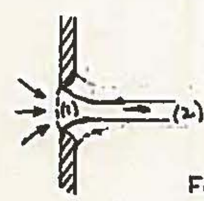


Fig. 5-6

Example 5-2. An arrangement of pipe and nozzle, forming the <u>syphon</u> indicated in Fig. 5-7, is provided for withdrawal of (fresh) water from a large open reservoir. Stream diameters and elevations at significant points in pipe or at nozzle are as indicated.

(a) Determine the rate of discharge via the nozzle if the flow were negligibly impeded by energy dissipation due to fluid friction and turbulence, and also if otherwise available information indicates that in various portions of the stream the energy dissipation may be expressed and evaluated as the following functions of local velocities.

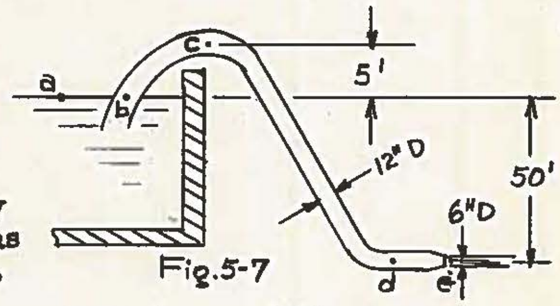


Fig. 5-7

Portion of stream	$a \geq b$	$b \geq c$	$c \geq d$	$d \geq e$
ϕ in that portion	$(\bar{u}_{pipe})^2/2$	$2(\bar{u}_{pipe})^2/2$	$5(\bar{u}_{pipe})^2/2$	$0.1(\bar{u}_{exit})^2/2$

Report also the ratio of the actual to the ideal rate of delivery, or what might be called the <u>coefficient of discharge</u> for the system:

(b) Determine the pressure at point <u>c</u> (both gage and absolute) under the actual flow conditions, for the purpose of anticipating if that pressure might be sufficiently low that the flow will be obstructed by the release and collection of vapor of dissolved gases at that point.

(c) Determine and tabulate the geopotential, kinetic and flow-work energies and their aggregate (Σ), and the associated pressure, at the significant points from reservoir surface to exit jet; also the progressive energy dissipation accompanying the flow.

Through these data exhibit graphically, to an abscissa representing distances along the pipe and an ordinate of energy-per-unit-mass (or "head"), the individual values and progressive aggregates of the geopotential, flow-work and kinetic energies, and of the energy dissipation.

Solution.

(a) For the idealized flow conditions, the kinetic energy at reservoir surface is negligible and the pressures both there and at nozzle exit are atmospheric and thus equal. Thus sufficient data are available for proceeding. Regarding the exit plane as a reference plane, and writing the corresponding energy equation

$gz_a (= 32.17 \times 50) = u_e^2/2$, or $u_e^2/2 = 1{,}609$ ft-lbf/slug and $u_e = 56.7$ ft/sec. Thus $\dot{m}$ ideal $= \rho u_e A_e = 1.94 \times 56.7 \times \pi (6/12)^2/4 = 1.94 \times 11.1$ (cfs) $= 21.5$ slugs/sec, or 695 lbm/sec.

For the actual flow conditions, ϕ (aggregate) $= (1 + 2 + 5)(u^2_{pipe}/2) + 0.1\, u_e^2/2$ and by introducing the continuity equation,

$$\phi = \left[8 \times (6/12)^4 + 0.1\right] u_e^2/2 = 0.6\, u_e^2/2.$$

Thus, by the energy equation but now with inclusion of $_a\phi_e$,

$gz_a (= 1609) = u_e^2/2 + 0.6\, u_e^2/2$; $u_e^2/2 = 1006$ ft-lbf/slug; and $u_e = 44.85$ ft/sec. Thus $\dot{m} = 1.94 \times 44.85 \times \pi (6/12)^2/4 = 17.1$ slugs/sec or 550 lbm/sec; and the coefficient of discharge $= 550/692 = 0.795$.

The velocity in the pipe, at sections b, c and d, will thus have been $\left[44.85 \times (6/12)^2\right] = 11.2$ ft/sec, with corresponding kinetic energy of 62.8 ft-lbf/slug.

(b) By writing the energy equation between sections a and c,

$$gz_a + 0 + 0 = gz_c + u_c^2/2 + p_c/\rho + {}_a\phi_c$$

$$1609 = 1770 + 62.8 + p_c/\rho + 3(62.8)$$

$p_c/\rho = -412$ and $p_c = -799$ psf gage, -5.6 psig or 9.1 psia

The indicated vacuum of 5.6 psi or absolute pressure of 9.1 psi may be such as to cause release and collection of dissolved air at point c, and flow stoppage. Note in this connection that the pipe will need initially, but independently, to have been filled with water in order to have established at c the pressure deficit (ideally, -2.17 psig) required for initiating the outflow.

(c) When indicating herewith the successive conditions through-out the system, both the nominal and the total magnitudes of p/ρ and the gage and absolute pressures will be reported.

	gz	$\frac{u^2}{2}$	p/ρ $= \Sigma_a - (gz_n + u_n^2/2 + {}_a\phi_n)$	Σ $= \Sigma_a - {}_a\phi_n$	p
a	1609	0	0.0, nominal	1609, nominal	0.0
			${}_a\phi_b = 62.8 \frac{ft\ lb_f}{slug}$		
b	1609	62.8	$1609 - (1609 + 62.8 + 62.8) = -126$	1546	−1.69 psig
			${}_a\phi_c = 3(62.8) = 188.4$		
c	1770	62.8	$1609 - (1770 + 62.8 + 188.4) = -412$	1421	−5.6 psig
			${}_a\phi_d = 8(62.8) = 502$		
d	0	62.8	$1609 - (0 + 62.8 + 502) = 1044$	1107	14.1 psig
			${}_a\phi_e = 502 + 0.1(1006) = 603$		
e	0	1006	$1609 - (0 + 1006 + 603) = 0$	1006	0.0

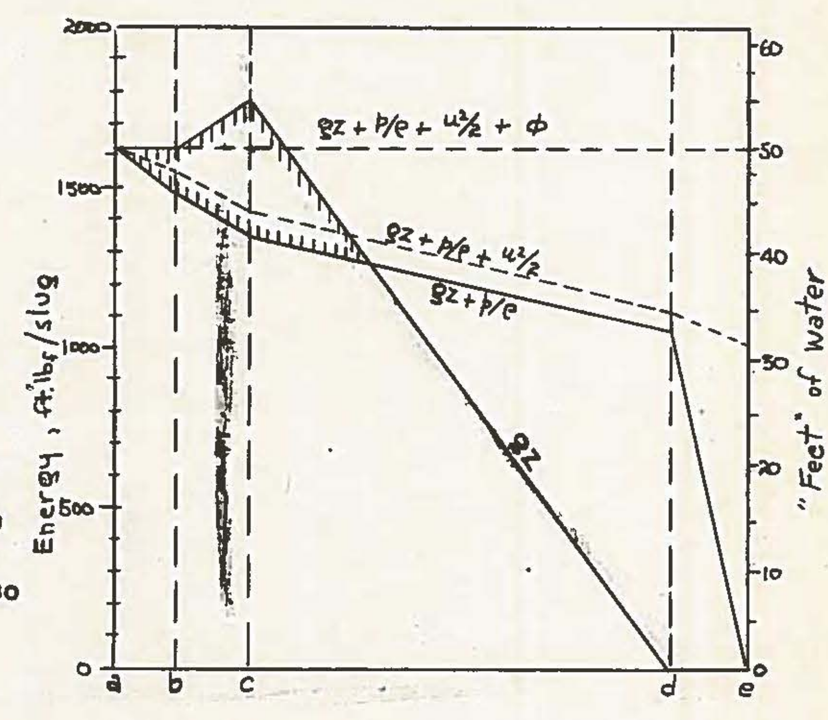


Fig. 5-8

Figure 5-8 provides the graphical representation of these findings. Note that the vertical position of the line for gz + p/ρ, relative to the horizontal axis, represents the "piezometric head" (art. 5-8) at any section; also that the hatched portions of this and the line for gz indicates that portion of the system in which the pressure is less than atmospheric and environmental air in-leakage or air release from solution is of concern.

In hydromechanic literature the line for gz + p/ρ, is known as the <u>hydraulic gradient</u>, and that for gz + p/ρ + $u^2/2$ as the <u>energy gradient.</u> The vertical distance of the latter below the upper (horizontal) line evidently represents the progressively increasing magnitude of the energy dissipation

(ϕ). When drawn in practice, distances along the abscissa may represent to scale either lineal distances along the pipe line or their horizontal projection, as either is the more significant.

5-12. Momentum Equation. It is evident that the facilities provided by the energy and continuity equations make them most useful engineering tools. However, they are not sufficient to allow specific analysis of the forces which are involved in producing any change in the state of motion of a fluid stream. For example, the energy and continuity equations suffice to calculate the necessary amount of work input to a stream to produce a given effect, but not to determine the manner of obtaining the forces required to produce the effect.

A momentum equation, developing directly from Newton's 2nd Law, provides a primary tool in such investigations. The equation is in effect simply such an adaptation of that Law as will make it more convenient in analyses of fluid flow than in those of solid-body motions. That is, writing it in a differential form permitting its application to a fluid mass δm* which is momentarily located within an infinitesimal length of stream tube, but is displaced by an equal mass in time dt, $df = \delta m\, da$ or $\delta m(du/dt)$. But, as the mass-rate of flow in the tube ($\dot{m}$) is $\delta m/dt$, the relation may be written in the alternative form $df = \dot{m}\, du$.

However, both force and velocity have directional significance and are thus vectorial. This aspect is introduced by writing further that $\partial f_x = \dot{m}\, \partial u_x$ or $\partial f_y = \dot{m}\, \partial u_y$, where the subscripted items denote force and velocity components in x- and y-directions, and the partial differential notation (∂) has the same connotation. In fluid-mechanics the great advantage of these adaptations is that, for steady flow and thus at constant $\dot{m}$, they are directly integrable along a stream tube to the forms

$$f_x = \dot{m}(u_{2,x} - u_{1,x}), \text{ or } f_y = \dot{m}(u_{2,y} - u_{1,y}) \qquad (5-7)$$

The resultant of these forces will be their vector sum.

The product mu being a vector quantity known as the momentum of mass m,

---

* - The symbol denotes a small quantity of an item having no directional significance, such as that of mass or of energy.

the relation states that the force required to change the velocity of a stream having a mass-rate of flow $\dot{m}$ equals the rate of change of its momentum, or at times abbreviated to the change of momentum-rate. The oppositely directed but equal restraint with which the stream resists change of its motion may be described as its inertial resistance, with the direction-reversal denoted by symbols such as $-F_x$ or $-F_y$.

Sketch (a) of Figure 5-7 depicts these considerations as they relate to a differential length of stream tube across any cross-sectional area of which effectively the same velocity exists, although along which the velocity changes.

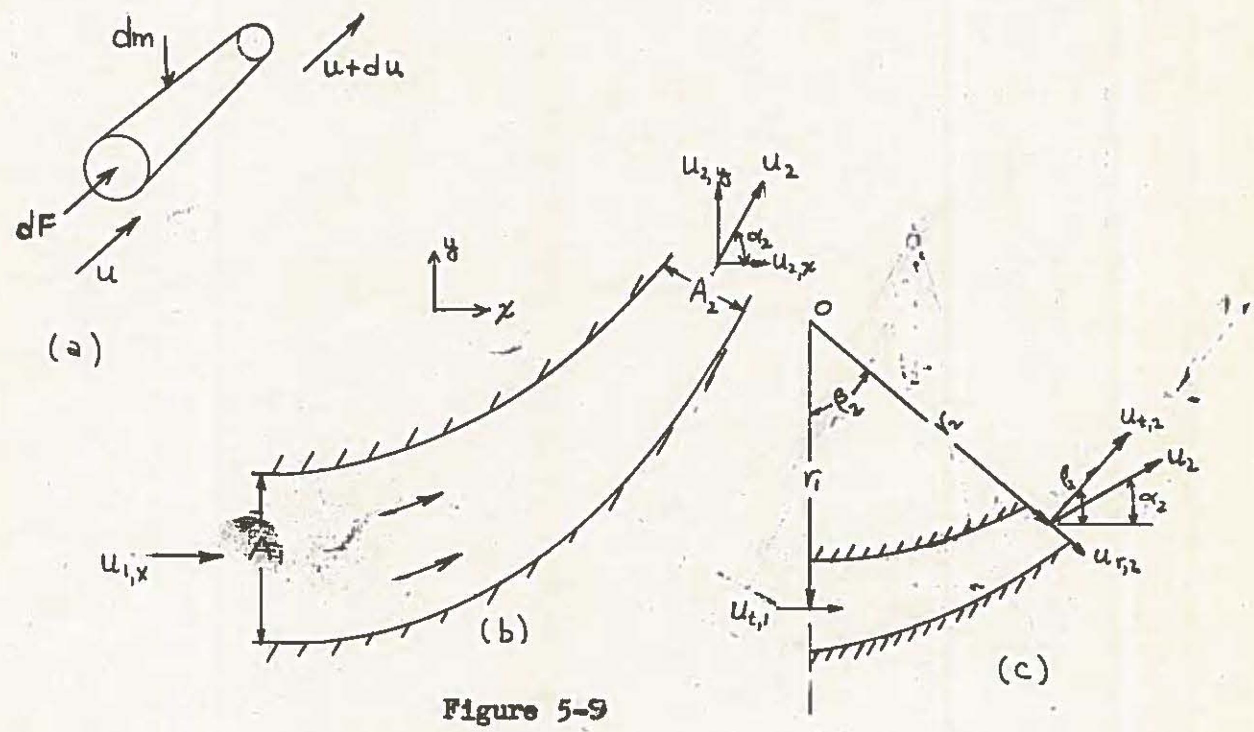


Figure 5-9

For utilization of these relations in connection with the finite streams of real fluids with which the engineer is concerned, it will in principle be necessary to summate the forces attributable to the individual stream tubes which ideally or nominally comprise the full stream. It will later be seen that, in a curving stream such as that of sketch (b) of Figure 5-9, the individual tube velocities change notably with their transverse position in the stream, and such force summations might well become quite complex and bothersome, or of uncertain validity. However, if sections (1) and (2) are taken as ones sufficiently removed from the region in which directional change is occurring, mean velocities at those sections ($\bar{u}_1 = \dot{m}/\rho A$) may be used with generally minor approximation, so justifying the relations

$$F_x \ (=\Sigma f_x) = \dot{m}(\bar{u}_{2,x} - \bar{u}_{1,x}), \text{ and } F_y = \dot{m}(\bar{u}_{2,y} - \bar{u}_{1,y}) \qquad (5\text{-}8)$$

In deriving the foregoing equations, attention was given only to the behavior of the fluid particles occupying the stream channel between sections 1 and 2. This group of particles is the equivalent of the familiar <u>free body</u> of solid-body mechanics. Only a single, all inclusive force was defined; that which caused a change in the momentum rate of the fluid. This force, then must be the resultant of all external or <u>body forces</u> acting on the fluid, whatever their origin. Examples of these are pressure forces, gravitational forces and <u>wall forces</u>. The last denotes those applied to the fluid by the walls of the confining channel. From the action-reaction principle, there is generally acknowledged the equal but oppositely directed counterpart of the wall force, namely the <u>reaction force</u> of the fluid on the channel.

In analyses of the many rotating types of engineering equipment a more immediate interest and utility may attach to the <u>moment of a force</u>, or the torque, about some axis. In this situation attention may better be directed to velocity and force components directed (a) normal to and (b) colinear with a radial line from a representative point in a stream to a selected (polar) origin. Subscripts t and r will distinguish the tangential and radial components of these. Sketch (c) of Fig. 5-9 illustrates, where 0 is the origin, and r_1 and r_2 are radial distances to points in the stream to which mean velocities $\bar{u}_1$ and $\bar{u}_2$ may be ascribed.

By analyses paralleling those relating to the above force components (F_x and F_y), it develops that the moment of the tangential component of their resultant, about origin 0, may be expressed as

$$\text{Torque } (q) = \dot{m}\ (\bar{u}_{t,2}\ r_2 - \bar{u}_{t,1}\ r_1). \qquad (5\text{-}9)$$

As a product of the form <u>m u r</u> is known as a moment of momentum or <u>angular momentum</u>, this relation may be regarded as saying that the torque operating on a stream having a stated mass-rate of flow but effecting change in velocity, equals the <u>rate of change of its angular momentum</u>, or be abbreviated to the change of angular-momentum-rate.

Example 5-3. Regarding the following data as ones applying to a curved nozzle such as is represented in sketches (a) and (b) of Fig. 5-9, determine F_x, F_y and q if the stream is fresh water.

Data:- A = 6 sq. in.; $\alpha_1 = 0.0^o$; $u_1 = u_{x,1} = 20$ ft/sec.;
A = 2 sq. in.; $\alpha_2 = 30^o$; $r_1 = 2.60$; $r_2 = 3.25$ ft.
$\beta_2 = 65^o$

Solution:-

$\dot{m} = \bar{u}_1 A_1 \rho_1 = 20 \times (6/144) \times 1.94 = 0.833$ (cfs)x1.94 = 1.617 slugs per second, or 52.0 lbm/second.

$u_2 = 20 \times (6/2) = 60$ ft/sec; $u_{2,x} = u_2 \cos 30^o = 52.0$ ft/sec;

$u_{2,y} = u_2 \sin 30^o = 30.0$ ft/sec.

$F_x = 1.617(52 - 20) = 51.7$ lbf; $F_y = 1.617(30 - 0) = 48.5$ lbf;
$F = \sqrt{51.7^2 + 48.5^2} = 70.9$ lbf.

$u_{t,2} = u_2 \cos(\beta_2 - \alpha_2) = 60 \cos 35^o = 49.1$ ft/sec.

$q = 1.617(49.1 \times 3.25 - 20 \times 2.60) = 174$ lbf-ft.

5-13

Problems -

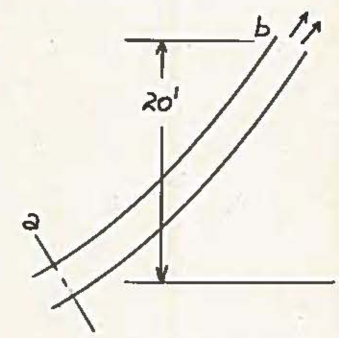
1. A 5 lb mass at a location where g = 32.17 ft/sec is raised against gravity a distance of 10 ft. What is its mass in slugs and its weight in pounds and in poundals; what will be the energy required to have lifted it the specified distance and the resulting increase in the geopotential energy stored in the elevated position, expressing these in ft. lb/lb, and also in foot-poundal per pound and per slug and foot-pounds per slug?

2. A 5 lb mass is permitted to fall without resistance of any character through a vertical distance of 10 feet at a location where g = 32.17 ft/sec . What will be the velocity acquired, in ft/sec, and the corresponding increase in kinetic energy, in ft-lb per pound and per slug? Ans. 25.4 ft/sec.

3. Five (5) pounds of a liquid of a density of 60 lb/cu ft are being delivered along a pipe line of 1.44 sq in area in which the pressure is 10 lb/in . What force on the after "face" of this parcel of fluid would be required to effect its motion against the resisting pressure, through what distance would this force have to act in order to have effected the entry of the mass into the space

initially ahead of it, what work energy would so have been required to have effected the entry of the mass, in ft/lb and in foot-pounds per pound and per slug, and what would be the magnitude and significance of the quotient p/ρ in this situation?

4. (a) For the pipe shown in the figure what pressure (lb/sq in gage) will be required at point a to effect flow of water upward through the pipe and into the atmosphere at the rate of 100 gpm, the pipe being of 2" inside diameter and frictional resistance producing an E_f of 3 ft. lb/lb? Ans. 10 psig.



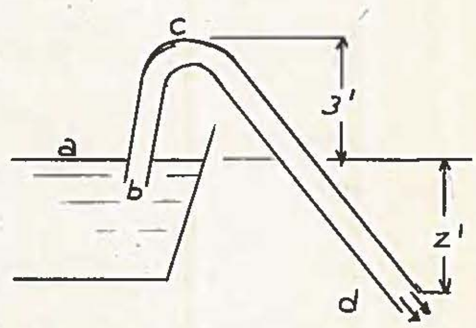
Prob. 4

(b) What pressure would be required to maintain delivery at the same rate if a nozzle were installed at the delivery end of the pipe, the nozzle producing a jet of 3/4" diameter and imposing an additional E_f of 1.0 ft. lb/lb?

Ans. 45.8 psig.

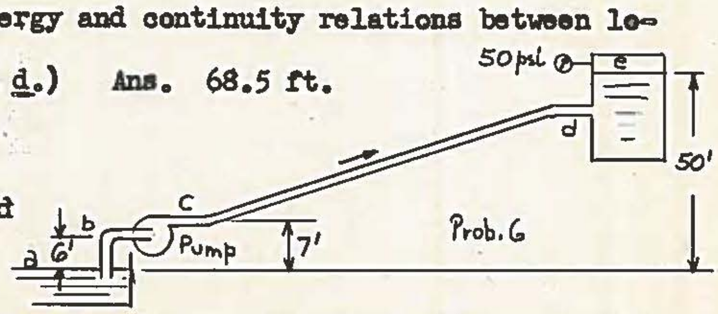
(c) How would the above pressures be influenced if the fluid were oil of a specific gravity of 0.8, all other data remaining the same? Ans. 8.0 psig; 36.6 psig.

5. To avoid vapor and air collection at the top of the siphon shown –, with consequent "vapor-binding" and stoppage of the flow, it is considered that the vacuum at point c should not exceed 27 inches of mercury. The pipe diameter is 3" and the magnitude of E_f between a and d may be taken as $4(u_p^2/64.35)$ ft.lb/lb, one fourth of it occurring between a and c. What may be the corresponding maximum magnitude of Z, and consequent maximum rate of outflow obtainable by use of the siphon? (Proceed by writing simultaneous energy and continuity relations between locations a and d and locations c and d.) Ans. 68.5 ft.



Prob. 5

6. For the installation indicated compute the power required for operating the pump if oil (sp.gr. = .85) is to be delivered from the lower reservoir to the upper tank



Prob. 6

at the rate of 200 gpm? The reservoir is at atmospheric pressure, and the delivery tank at 50 psi gage. The pipe lines are of 3" inside diameter. Energy dissipations by friction and turbulence may be taken to be -

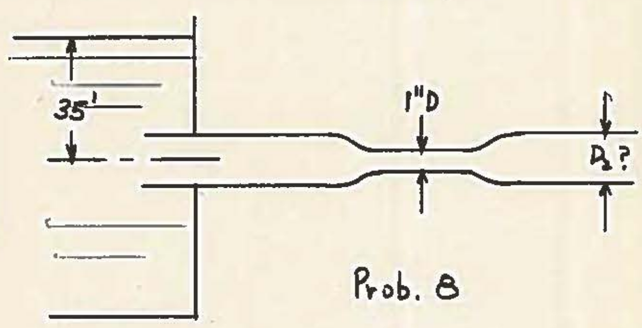
$_a(E_f)_b = 3(u_p^2/64.35)$; $_b(E_f)_c = 0.35$ x pump work input per pound;

$_c(E_f)_d = 10(u_p^2/64.35)$. The kinetic energy existing at d will undoubtedly be entirely dissipated on entering the delivery tank.

Determine also the pressures at points b and c. Ans. 13.4 HP

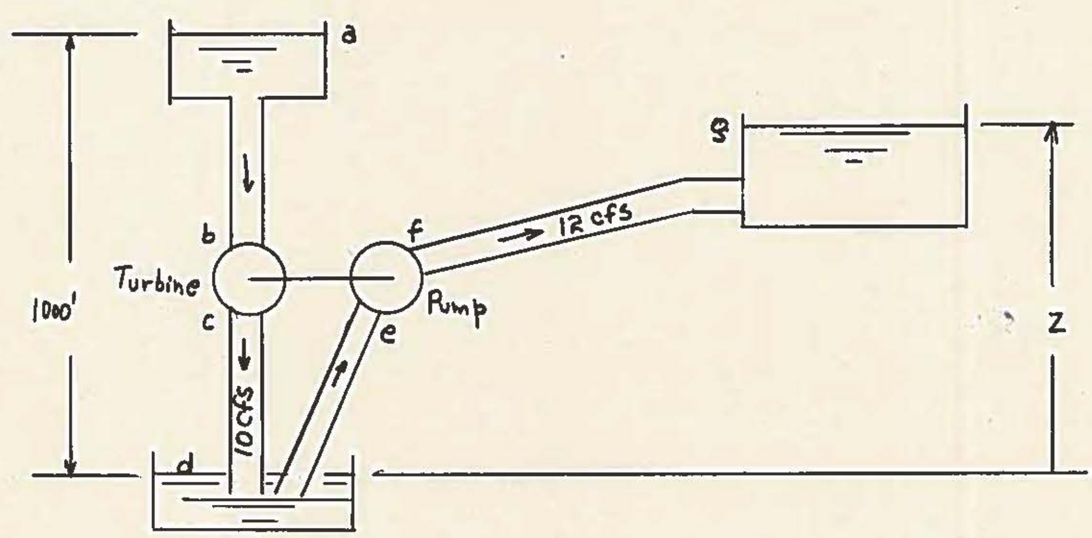
7. A 50,000 HP hydro-electric plant is contemplated using water from a lake 500 ft above the plant site. If the best turbine-generator efficiency to be expected is 92% how much water must be available? Ans. 959 cfs.

8. Calculate the diameter D_2 which will ideally allow a velocity of 10 ft/sec at the discharge end of the pipe. Assume a vapor pressure of 1.5 psia. Ans. 2.55"



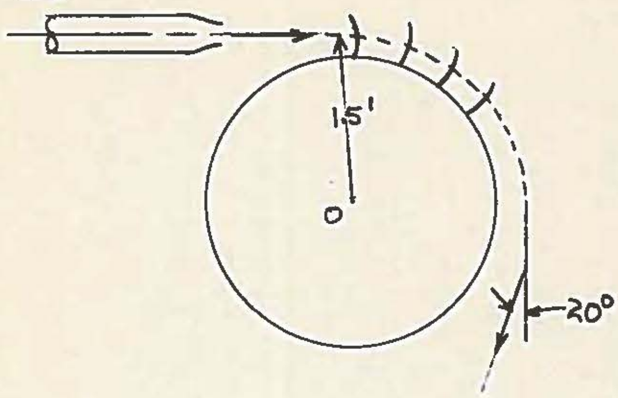
Prob. 8

9.



Under the conditions indicated in Fig. 5 calculate the height, Z, to which the system is capable of pumping water. Ans. 566 ft.

10.



Water is injected against a 3' diameter water wheel, rotating in a horizontal plane with a velocity of 80 ft/sec and a rate of 10 cfs and discharges at an angle of 20° to the vertical with a velocity of 10 ft/sec.

Calculate the torque developed about O. Ans. 2037 ft lb_f

CHAPTER 6

ENERGY DISSIPATION ACCOMPANYING FLOW, IN ENCLOSING CHANNELS

6-1. Foreword. In the energy discussions of the preceding chapter it became evident that, in practical analyses of any flow phenomena, consideration must inescapably be given to the dissipation of mechanical energies into disorganized molecular activity (perhaps with concurrent energy departure as heat), in consequence of fluid friction and or resultant turbulence. The major purpose of the present chapter will be to recognize the factors which influence such dissipation, and the means whereby relevant experimental data or rational analyses may best be organized and utilized. But attention is here limited to situations in which the flow is through enclosing channels, such as pipes.

6-2. Factors Influencing the Energy Dissipation. The first step in the analysis of energy dissipation in flow phenomena is the recognition of the individual factors which, through experience or rationalization may be expected to influence its amount per unit mass of fluid (ϕ). These are found to include the following.

(a) The extent of the surface along which the fluid passes;

(b) Relevant characteristics of the flowing fluid;

(c) Features associated with the character of the flow past those surfaces; and

(d) Features associated with the character of the surfaces past which the flow occurs.

Immediate attention will be limited to channels of uniform size and shape, and to those portions in which steady and uniform flow of a given character has been established and maintained. Such a channel would be exemplified by ordinary piping in a section adequately distant from any disturbing influence. In such circumstances the energy dissipation may properly be expected to be directly proportional to the linear distance travelled, and it therefore becomes suitable to give attention to the dissipation per unit mass and per unit distance, or ϕ/L.

Returning to consideration of the above factors:-

(a) The amount of surface along which the fluid stream passes is evidently expressible, per unit mass of fluid, by the ratio

$$\frac{\text{perimeter of channel x length}}{\text{(area of stream x length) x density of fluid}} \quad \text{or} \quad \frac{1}{\rho} \Big/ \frac{\text{area}}{\text{perimeter}}$$

The ratio area/perimeter is known as the hydraulic radius of channel and stream, and here denoted by symbol r_h, with the above ratio so becoming $1/(\rho r_h)$. For a circular channel if running full, $r_h = \frac{\pi d^2/4}{\pi d} = \frac{d}{4}$ or $r/2$, where r is its actual radius. In the following major attention is given to circular channels, and their diameter rather than the hydraulic radius is employed as a size index.

(b) An evident and relevant characteristic of the fluid is its density, and an influence of that has just been noted. The further characteristic influencing the energy dissipation is the dynamic viscosity (μ, art. 1-8). When some attention is later given to the flow of gases, the influence of temperature on density and that of pressure (i.e., the compressibility) will require attention.

(c) The only characteristic of the flow requiring immediate recognition is the velocity, normally considered as the mean velocity $\bar{u}$ (i.e., $\dot{m}/\rho A$).

(d) Features associated with the character of the surfaces confining the stream, and along which the flow occurs, may in general be described only in qualitative terms such as the channel shape (λ, lambda) and roughness of surface (ϵ, epsilon)

These include all variables that need now be considered. On survey of them, it appears that they fall into two distinct categories, one capable of direct quantitative evaluation and the other permitting only rather qualitative description. That is -

Quantitatively expressible variables	Qualitatively expressible variables
Length of channel, L	Shape of channel, λ
Diameter (d) if circular, or the hydraulic radius, r_h	Roughness of surfaces, ϵ
Density of fluid, ρ	
Viscosity of fluid, μ	
Velocity of stream, $\bar{u}$	

Because of the qualitative character of shape and surface roughness, it is generally practicable to provide definite relations associating the energy dissipation only with the quantitative variables influencing it. Also, except in simpler circumstances, it will be necessary generally to content oneself with experimentally-obtained information concerning their interdependence as found for channels of similar shape and roughness. An exception is considered in the following article.

6-3. Laminar Flow in Circular Ducts. For analysis of the laminar flow of art. 5-2, but occurring in a circular duct of radius R such as represented in Fig. 6-1, refer in that figure to a cylindrical segment of the stream of length **L** and radius **r**, and having a motion relative to its environs which is governed entirely by the viscous restraint of art. 1-8. For maintenance of this

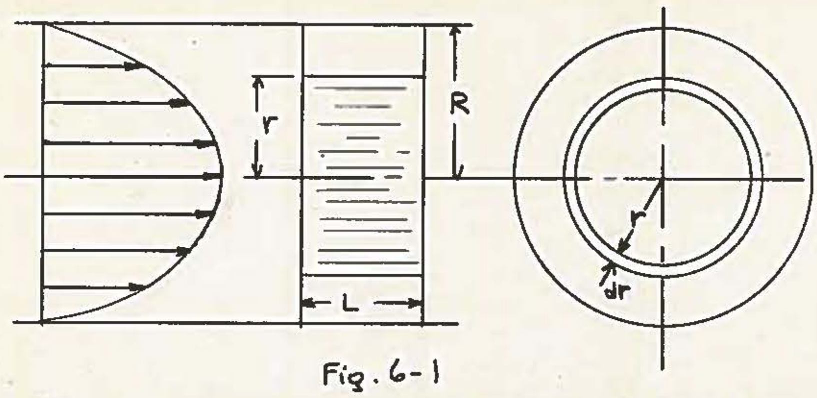


Fig. 6-1

motion a pressure excess and consequent force excess, $(p_1 - p_2)\pi r^2$, must exist as between up-stream and downstream faces of the cylinder, in amount such that

$$(p_1 - p_2)\pi r^2 = \Delta F = (2\pi r L)\tau \qquad (6\text{-}1)$$

where τ is the resistance per unit area which opposes the motion and is a function of the viscosity of the fluid. But, by eq. 1-4,

$$\Delta F = -(2\pi r L)\mu \frac{du}{dr}$$

where du/dr is the rate of deformation or velocity gradient in the lamina at the surface of the cylinder.

Thus, $du = -\left(\frac{p_1 - p_2}{2\mu L}\right) r\,dr$, or $-\left(\frac{p_1 - p_2}{4\mu L}\right) d(r^2)$

As the pressure (or more exactly, $p + \rho z g$) must be the same throughout a transverse section of the stream, else transverse currents would occur, this relation may be integrated to

$$u, \text{ at radius } r, = -\left(\frac{p_1 - p_2}{4\mu L}\right)(r^2 - C^2)$$

where the constant of integration (C) is established by noting that u = 0 immediately at the duct surface, or where r = R.

Correspondingly,

$$u, \text{ at radius } r, = \left(\frac{p_1 - p_2}{4\mu L}\right)(R^2 - r^2) \qquad (6\text{-}2)$$

This relation is recognized as that of a parabola. The corresponding parabolic ~~distribution of velocity~~ is indicated to the right in the figure.

Also, $\dot{m} = \rho\int_0^R u(2\pi r\,dr) = \pi\rho\int_0^{R^2} u\,d(r^2)$, and

$$= \pi\rho\left(\frac{p_1 - p_2}{4\mu L}\right)\int_0^{R^2}(R^2 - r^2)\,d(r^2)$$

$$= \pi\rho\left(\frac{p_1 - p_2}{8\mu L}\right)R^4; \qquad (6\text{-}3)$$

and $\bar{u} = \frac{\dot{m}}{\rho\pi R^2} = \left(\frac{p_1 - p_2}{L}\right)\frac{R^2}{8\mu}$, (6-4)

or $= \left(\frac{p_1 - p_2}{L}\right)\frac{D^2}{32\mu}$.*

By combining equations 6-2 and 6-4,

$$u = 2\bar{u}\left(1 - \frac{r^2}{R^2}\right) \qquad (6\text{-}4a)$$

From this it is apparent that the maximum velocity, which exists at the center of the pipe has twice the magnitude of the mean velocity $\bar{u}$.

As $\phi = \frac{p_1 - p_2}{\rho}$ for steady uniform flow, this may be written as

$$\bar{u} = \frac{\rho \phi D^2}{32 L \mu} \tag{6-5a}$$

or rearranging in terms of dimensionless parameters

$$\frac{\phi D}{L \bar{u}^2} = 32 \left(\frac{D \bar{u}}{\mu/\rho} \right)^{-1} \tag{6-5b}$$

The ratio $\frac{\mu}{\rho}$ in this relation is recognized as the <u>kinematic</u> viscosity (ν) of art. 1-8. To verify the non-dimensionality of the above combinations, by reference to table 1-2, $\frac{\phi D}{L \bar{u}^2}$ has the basic dimensions $\frac{(L^2 T^{-2}) L}{L (L^2 T^{-2})}$, and $\frac{D \bar{u} \rho}{\mu}$ the dimensions $\frac{L (L T^{-1})(M L^{-3})}{M L^{-1} T^{-1}}$, or thus $M^0 L^0 T^0$ for both.

In some instances it will be advantageous to have a parallel relation in terms, however, of a <u>drag</u> force (F_{drag}) acting along the walls of the channel which confines the flow. Recognizing this force as the force ΔF of eq. 6-1 at $r = R$,

$$F_{drag} = (p_1 - p_2) \pi R^2, \text{ and by eq. 6-4,}$$
$$= 8 \pi \mu L \bar{u};$$

or again arranging in terms of dimensionless parameters, by introducing the product ($\bar{u} \rho D$) on both sides of the relation,

$$\frac{F_{drag}}{\rho D L \bar{u}^2} = 8 \pi \left(\frac{D \bar{u}}{\mu/\rho} \right)^{-1} \text{ and also } \doteq \frac{\pi}{4} \frac{\phi D}{L \bar{u}^2} \tag{6-6}$$

Alternative manners of expressing these parameters are frequently convenient, in the feature of not requiring direct consideration of the velocity. They are obtained by employing the item defined (art. 1-14) as the mass-rate

---

* - An error in regarding $\bar{u}^2/2$ as the mean kinetic energy per unit mass of fluid, when a stream is advancing with this manner of velocity distribution, is seen by writing a corresponding <u>rate</u> of such energy passage. That is, $\frac{\bar{u}^2}{2}(\rho \bar{u} \pi R^2) = \pi \rho \bar{u}^3 \frac{R^2}{2}$

In contrast, the true rate is $\sum (\frac{\bar{u}^2}{2}) \delta \dot{m}$. Expressing this summation as follows, $\sum (\frac{\bar{u}^2}{2}) \delta \dot{m} = \int_0^R (\frac{\bar{u}^2}{2}) \rho u (2 \pi r dr)$

and, from eq. 6-4a

$$= 8 \pi \rho \bar{u}^3 \int_0^R \left(1 - \frac{r^2}{R^2}\right)^3 r\, dr = \pi \rho \bar{u}^3 R^2$$

That is, the true kinetic energy is twice that computed on the basis of mean velocity $\bar{u}$. But it was noted in art. 5- that the velocities at which laminar flow may persist are in general so low that corresponding kinetic energies are negligible except in precise energy accounts.

of flow per unit area of channel, known variously as the flow-density or the mass-velocity, and represented by the symbol G . That is, $G = \dot{m}/A = \bar{u}\rho$, or $\bar{u} = \frac{G}{\rho}$. Introducing this expression for $\bar{u}$, eq. 6-5 and 6-6 become

$$\frac{F_{drag}\,\rho}{D\,L\,G^2}, \text{ or } \frac{\pi}{4}\frac{\phi D \rho^2}{L G^2}, = 8\pi\left(\frac{DG}{\mu}\right)^{-1} \qquad (6\text{-}7)$$

A noteworthy observation with reference to these last relations is that in each a parameter involving the energy dissipation or the drag force is seen to be a function of a common parameter $\frac{D\bar{u}\rho}{\mu}$, or $\frac{D\bar{u}}{\nu}$. This is later found to have marked utility in a variety of connections other than the present.

6-4. Geometric Shape and Similarity. In the simple situation above it was practicable to relate the energy dissipation accompanying flow to the variables influencing it. In other situations, where a different shape of channel, roughness of its wall surfaces or greater stream velocities may have contributed in causing turbulent flow, the provision of equivalent analyses may well be quite formidable, or of uncertain propriety if attempted. But it is most desirable that some means be available whereby a correlation between relevant variables may still be provided in a useful form. Articles 6-7 and 6-5 indicate the means employed to such purposes, and the reasoning behind associated procedures.

A first observation in such connections is that, except as they may be represented by diagram or picture, even simpler shapes (such as polygons) may be described only in terms of ratios between the lengths and the angular orientations of the sides of the figure. Note that such ratios are non-dimensional, but also that a smaller and a larger figure may be said to be geometrically similar if the magnitudes of such ratios are the same for both.

Parallel considerations apply to the concept of the relative roughness of a surface, this being in effect an index of the (dimensionless) ratio between the size of surface irregularities and the size of the channel formed by those surfaces. Larger and smaller channels exhibiting identity in both the shape ratios and the roughness ratios would be described as geometrically similar in all presently significant respects.

It is not unreasonable to anticipate that the pattern of the flow occurring even in geometrically similar channels may well be quite different in character under different conditions. Even in a given channel the flow under some conditions may be laminar, but turbulent under others. But neither is it unreasonable to accept a possibility that a similarity also in the flow pattern, or a dynamic similarity, might be attained in flow through geometrically similar channels if certain further specifications were made.

6-5. Dynamic Similarity; the Reynolds Index. The notion of a possible similarity in the flow pattern exhibited by a given fluid, or even by various fluids, when enroute through geometrically similar channels, was suggested above. Geometric similarity signifies only that still photographs of shapes which are geometrically similar but of different size will be identical if made to properly related linear scales. Similarly, motion pictures of dynamically similar flow in geometrically similar channels would be identical if made to properly related linear scales and if taken at properly related speeds.

For recognizing a basic criterion by conformity with which dynamic similarity might be attained, note initially that the path of any element of a turbulent stream, if only viscous and inertial restraints are significant, would be established through joint exercise of their individual influences. Thus it is not illogical to anticipate that, if elements in several streams were to be constrained to follow similar characters of paths in different but geometrically similar channels, this dynamic similarity might be attained if at any given location the ratio between the inertial and the viscous restraints operating on the element were the same.

To formulate such a force ratio refer to fig. 6-2, in which motion in similarly curved paths by geometrically similar (cubical) fluid elements is represented as taking place enroute past geometrically similar obstructions in the two flow fields. Symbols with and without the prime (') will distinguish items relating to each. For simplicity only radial components of the inertial restraints will be considered. Recall that the pertinent forces are expressible as

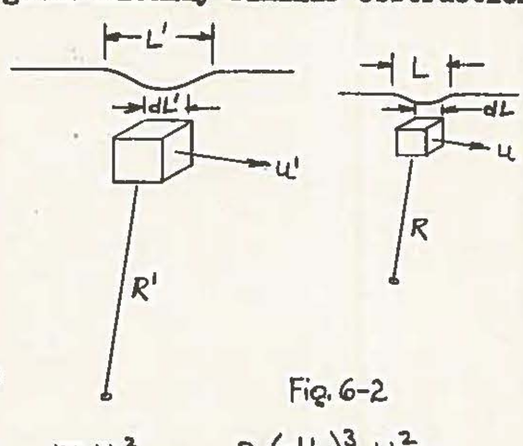


Fig. 6-2

$$\text{Inertial restraint (radial), } F_i, = \frac{mu^2}{R} = \frac{\rho\,(dL)^3\,u^2}{R}$$

$$\text{Viscous restraint, } F_\mu, = \mu\,(dL)^2\,\frac{du}{dL}$$

Thus the ratio $\frac{F_i}{F_\mu}$ equals $\frac{(dL)^2 u^2 \rho}{\mu R\, du}$ for the one element, and $\frac{(dL')^2(u')^2\rho'}{\mu' R' du'}$ for the other. For equality of these ratios, $\frac{(dL/dL')(u/u')\,(\rho/\rho')}{(\mu/\mu')(R/R')(du/du')} = 1.0$

. But for like flow patterns past the geometrically similar obstructions, $\frac{du}{du'} = \frac{u}{u'}$, $\frac{dL}{dL'} = \frac{L}{L'}$ and $\frac{R}{R'} = \frac{L}{L'}$. Therefore, for the composite dynamic and geometric similarity, and recalling the presumed force-ratio equality,

$$\frac{L}{L'}\,\frac{u}{u'}\,\frac{\rho}{\rho'}\,\frac{\mu}{\mu'} = 1.0 \; ; \; \text{or} \quad \frac{L'u'\rho'}{\mu'} = \frac{L u \rho}{\mu} \qquad (6\text{-}8)$$

Here the products of the form $L u \rho$, forming the numerators, may be regarded as indices of the inertial influence, with the denominators (μ) being indices of the viscous influence.

A verbal interpretation of these formulations is that

For flow through channels that differ in size (L) but are geometrically similar in shape and relative roughness, dynamically similar flow patterns result even with flow at different velocities (u), and/or with fluids of different densities (ρ) and viscosities (μ), if in each instance the index $L u \rho / \mu$ is the same.

It is significant that the function so evolved is again the dimensionless parameter encountered in art. 6-3, in the analysis of laminar flow through circular pipes, except as the symbol L here denotes any representative size dimension while previously the pipe diameter was the relevant size item. This parameter is known as the Reynolds Index (or Number), being so named in recognition of the eminent scientist Osborne Reynolds who first (1883) directed attention to its significance and utility.

As was noted in art. 1-11, a convenient feature of any dimensionless parameter such as the Reynolds index, or the energy-dissipation or drag force parameters of equations 6-5, 6-6, or 6-7, is that, for given real magnitudes of relevant items such as size, velocity, et cetera, the paramter would exhibit the same numerical magnitude in any system of consistent units; that is, ones involving the same proportionality constant (normally unity) in the force-mass-acceleration relation of the second law of motion (art. 1-3). A conventional practice is, therefore, to employ consistent systems, although departures have been made in engineering practice. The accompanying table supplements Table 1-1 in indicating a number of consistent combinations.

Table 6-1

Item	Units		
size	foot	foot	centimeter
velocity	ft/sec	ft/sec	cm/sec
mass	slug	pound	gram
density	slug/ft	lb/ft	gram/cm
force	pound	poundal	dyne
pressure	lb/ft	poundal/ft	dynes/cm
specific energy	ft.lb/slug	ft.poundals/lb	ergs/gram
viscosity, absolute	slugs/sec.ft. or lb.sec./ft	lb/sec.ft; or poundal sec/ft	poises
viscosity, kinematic	ft /sec	ft /sec	cm /sec

6-6. Organization of Variables by Dimensional Analysis; the Pi (π) Technique.

In article 6-3 it was developed through detailed analysis that, for laminar flow through a circular duct, the energy dissipation per unit mass of fluid (or the drag force on the channel walls) might be represented by the relations

$$\left.\begin{array}{c} \phi D/L\bar{u}^2 \text{ ; or } \phi D\rho^2/LG^2 \\ \text{and} \\ 4F_{drag}/\pi DL\bar{u}^2\text{; or } 4F_{drag}\rho^2/\pi DLG^2 \end{array}\right\} = 32\left(\frac{D\bar{u}\rho}{\mu}\right)^{-1}, \text{ or } 32\left(\frac{DG}{\mu}\right)^{-1}$$

In the last article more abstract considerations were given to turbulent flow situations, these probably as induced by greater flow rates, steeper velocity gradients (d u/d R), channel shape and/or surface roughness, and as resulting from the snow-balling effects which initiate turbulence. And it developed that the numerical magnitude of the Reynolds index appearing to the right in the above equation might reasonably provide evidence also as to the dynamic nature of the flow in turbulent streams.

With this background we would be in a position now to to investigate the validity of the foregoing, as ascertained from experimental evidence, and to become familiar with the facilities employed for their practical utilization. But there is some advantage in surveying first several techniques whereby the physical variables involved in such analyses may readily be organized into desired dimensionless combinations by attention only to the mathematical requirement of dimensional homogeneity in any physical relation (art. 1-13). However, one should not infer that the mere ability, through such dimensional analysis, to collect a number of relevant physical variables into dimensionless groupings is in itself sufficient. One of these must be physically justifiable, as was the Reynolds index, as evidence of the dynamic character of the flow.

The procedures of dimensional analysis require that the nature, number and primary physical dimensions of all influential variables shall first be recognized. For closed-channel flow phenomena (except for the flow of expansible gases and especially for their flow at sonic velocities) it has been seen above that, referring also to Table 1-2, these include

(1) the energy dissipation per unit mass and unit length of channel, ϕ/L, with dimension LT^{-2}

(2) the channel diameter (D) or hydraulic radius (r_h), with dimension L

(3) the mean stream velocity, $\bar{u}$, of dimension LT^{-1}

(4) the fluid density, ρ, of dimension ML^{-3}

(5) the fluid viscosity, μ, of dimension $ML^{-1}T^{-1}$

and also the highly influential but inherently non-dimensional items of channel shape (λ) and surface roughness (ϵ).

Two related manners of procedure, known as the <u>product-of-powers method</u> and the <u>pi (π) method</u>, are available for effecting the desired organization of such a family of variables.

(a) <u>Product-of-Powers Method</u>. In this procedure it is acceptably taken that any relation associating such physical variables might conceivably be expressed as a product of (possibly) variable but dimensionless powers of the primary variables. To illustrate,

$$f(\phi/L, D, \bar{u}, \rho, \mu) = f\left(\frac{\phi}{L} D^a \bar{u}^b \rho^c \mu^d\right), \text{ and also } = f'(\lambda \text{ and } \epsilon)$$

The shape and roughness items are properly dissociated from the other variables, due to their inherent non-dimensionality. But this also establishes that, when writing the above in terms of the basic dimensions of the variables, any relation associating the variables must be such that

$$f\left[(LT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c(ML^{-1}T^{-1})^d\right] = f'(M^0L^0T^0),$$

It so appears that, for the requisite dimensional homogeneity in this equation,

for homogeneity in M, $c + d = 0$, or $d = -c$;

for homogeneity in T, $-2 - b - d = 0$, or $b = -2 - d$ and $c - 2$;

for homogeneity in L, $1 + a + b - 3c - d = 1 + a + (c - 2) - 3c + c$
$= a - c - 1 = 0$;
or $a = c + 1$

These have provided the means whereby the variables may be collected into suitably dimensionless groups, and the above relation may be restated in the form

$$f\left[\left(\frac{\Phi}{L}\right)D^{1+c}\bar{u}^{c-2}\rho^{c}\mu^{-c}\right] = f\left[\left(\frac{\Phi D}{L\bar{u}^2}\right), \left(\frac{D\bar{u}\rho}{\mu}\right)^{-1}\right],$$

$$\text{and } = f'(\lambda \text{ and } \epsilon)$$

But, now recognizing that the writing of the functions in terms of products-of-powers was only a device for enabling collection of the variables into dimensionless groups, that device may be discarded and the last relation put in the general form $\frac{\Phi D}{L\bar{u}^2} = f\left(\frac{D\bar{u}\rho}{\mu}\right) f'(\lambda, \epsilon)$.

(b) Pi (π) Method. This method of procedure, although more flexible and direct, is based on somewhat more abstract reasoning. That will not be developed here, but instead several rules governing its application will be stated and their use illustrated sufficiently for present purposes. The designation of the method derives from the practice of describing as "π's" any dimensionless parameters which are thereby evolved. The rules are as follow.

I. Denoting the number of dimensional variables influencing a physical phenomenon by the symbol m, and the number of fundamental dimensions involved by symbol n, the number of independent π's required in effecting the correlation of all variables equals (m - n.)

II. The number of variables initially selected for inclusion in an individual π shall be one greater than the number of fundamental dimensions involved, or n+1. One may subsequently be found to acquire zero exponent and so to disappear from that π, but this need not cause undue concern. *

Except that all variables must obviously have been selected for appearance in one or another of the π's, and again with one of these necessarily justifiable physically as an index of dynamic or like similarity in the phenomenon in question, their selection for inclusion in a particular π is guided by subsequent convenience. Those so selected may well be (n - 2) of the items for which data are known or specified, and a further <u>wanted but unknown</u> item. Their collection into a dimensionless grouping is made by procedures rather like those of the product-of-powers method, but frequently becomes little more than a matter of inspection.

To illustrate by organizing again, into the <u>two</u> prior parameters, the above family of <u>five</u> variables involving in the aggregate <u>three</u> primary dimensions, $\pi_1 = D\bar{u}^a\rho^b\mu^c = (L)(LT^{-1})^a(ML^{-3})^b(ML^{-1}T^{-1})^c$, and $= M^0L^0T^0$

and $\pi_2 = (\Phi/L)D^a\bar{u}^b\rho^c = (LT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$, and $= M^0L^0T^0$.

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* - However, if one is found to be incapable of incorporation in any dimensionless group, this may be regarded as mathematico-physical evidence either that the item is in fact not properly to be regarded as an influential one, or that the influence of others have been overlooked.

By inspection of the first for homogeneity in M, b = - c; in T, a = - c = b; and in L, 1 + a - 3b - c = 1 + b - 3b + b = 1 - b, and = 0. Thus b = - c = a = 1, and $\pi_1 = \frac{D\bar{u}\rho}{\mu}$ or $\frac{D\bar{u}}{\nu}$.

By inspection of the second it is seen that for homogeneity in M, c = 0 and the density so disappears from π_1 . By only cursory further inspection it develops that for homogeneity in T, b = -2 and, in L, a = - b - 1 = 1. Thus $\pi_2 = \frac{(\phi/L)D}{\bar{u}^2}$.

To illustrate the versatility of the π-technique, presume, for example that it is desirable to associate ϕ/L , d, ρ and μ in an alternative parameter π_3 ; or perhaps ϕ/L, $\dot{V}$, ρ and μ in a π_4 , where $\dot{V}$ is the volume-rate of flow. For evolving the first,

$$\pi_3 = (\tfrac{\phi}{L})D^a\rho^b\mu^c = (LT^{-2})(L)^a(ML^{-3})^b(ML^{-1}T^{-1})^c, \text{ and } = M^0L^0T^0$$

By inspection in T, c = - 2; in M, b = - c and = 2; and in L, a = - 1 + 3b + c = 3. Thus

$$\pi_3 = \frac{(\phi/L)\rho^2 D^3}{\mu^2}$$

This parameter may be seen to be simply the product $\pi_2\pi_1^2$. By virtue of such inclusion of the Reynolds index it becomes to that degree an indirect index of the dynamic character of the flow.

For evolving the second supplementary parameter (π_4),

$$\pi_4 = (\tfrac{\phi}{L})\dot{V}^a\rho^b\mu^c = (LT^{-2})(L^3T^{-1})^a(ML^{-3})^b(ML^{-1}T^{-1})^c, = M^0L^0T^0$$

By inspection (in M) b = - c, (in T) a = - c - 2 and = - 2 b, and (in L) 1 + 3 a - 3 b - c = 1 + 3a - (3a + 6) + (a + 2) = 0, or a = 3. Thus

$$\pi_4 = \frac{(\phi/L)\rho^5}{\mu^5}\dot{V}^3$$

This parameter may similarly be seen to be $.484\,\pi_2\pi_1^5$, to which would apply the same observations as made above concerning π_3 .

6-7. <u>Application to the Flow of Liquids in Pipes.</u> The probabilities noted in art 6-5 and the practical utility of the techniques described in the last article have been well verified through a great accumulation of experimental data. The tests have been ones in which the pressure drop, and associated magnitude of ϕ , are determined for measured lengths of pipe in which laminar or turbulent flow is fully established, and with a variety of liquids flowing at different rates through pipes of different size and wall roughness. Verification of the preceding expectations is provided by the findings that, to such extent as the relative roughness of the wall of a pipe may be appraised precisely, the same relations between the Reynolds index (or its equivalents) and the other parameters persist if the flow is through geometrically similar channels. A basis of roughness appraisal is noted below.

There is only one exception, in a range of increasing flow rates within which transition from laminar to turbulent type of flow occurs (or the converse). The flow conditions are critical in the sense of being both unstable and subject to various extraneous influences (such as pipe vibration) which may trigger-off or may delay the transition.

Although earlier investigators endeavored to express their findings by algebraic equations,* current practice is to represent <u>graphically</u> the relations existing between useful parameters, or π's. The coordinates employed may be limited to the Reynolds index (π_1) and a <u>friction factor</u> (f) which is defined as $\frac{(\phi/L)D}{\bar{u}^2/2}$, and is evidently twice π_2; or a more comprehensive system of rectangular and skewed coordinates may be employed for providing greater versatility in use of the graph. Fig. 6-3 illustrates the latter.

The use of $\bar{u}^2/2$ in such a "factor" originates in its significance as a measure of the kinetic energy per unit mass. From that viewpoint the factor appears quite suitably as a ratio between the kinetic or organized and the dissipated energies for flow through similar channels of like diameter/length ratio. But recall, that for numerical suitability if the energy dissipation were preferably expressed in the (inconsistent) terms of ft.lbf/lbm, or "feet of fluid" and here represented as ϕ', the friction factor would appear as $\frac{64.35(\phi'/L)D}{\bar{u}^2}$.

- -

*- Before the recognition of the greater utility of graphical correlations of suitable dimensionless parameters, the engineer could only endeavor to deduce from his experiments more or less simple <u>formulas</u> for associating more evident variables. For the flow of water in pipes these ultimately took the form of several exponential equations which, from present viewpoints, resolve in effect into ones which might be hoped to express sufficiently the evidence provided by the entire family of curves of fig. 6-3. For example, $f = a(N_R)^n$, where <u>f</u> is again the "friction" factor, $\frac{2(\phi/L)D}{\bar{u}^2}$; and N_R is the Reynolds index or "number", $D\bar{u}/\nu$.

For the laminar-flow regime this is quite suitable, as the analytical and experimental evidence of both equation 6-5b and the figure indicate a constant value of 64.0 for <u>a</u>, and of -1.0 for <u>n</u>; or in alternative form, $\phi = \frac{32\nu L\bar{u}}{D^2}$. Comparably, at the greater magnitudes of N_R in the turbulent-flow regime, <u>f</u> is seen in the figure to become a constant determined by the relative roughness of the pipe, or alternatively, $n = 0$ and $N_R^{\,n} = 1.0$. Thus a suitable formula for these portions of the turbulent-flow regime is $\phi = \frac{aL\bar{u}^2}{2D}$, with <u>a</u> fixed by relative roughness.

At the more moderate values of N_R in the turbulent-flow regime, and many flow conditions lie in this range, no simple formula is capable of expressing the relationships indicated in the figure. The devising of a more complex but adequate one would become most difficult, or even impracticable. But, <u>if</u> for this portion of the figure one might reasonably allot an <u>average</u> value of some item (as, for example, n = -0.2), a resulting expression would be

$$\phi = \frac{La}{2}\nu^{-n}\bar{u}^{(2+n)}/D^{(1-n)}.$$

Formulas of this character, but presumptively suitable for all turbulent flow conditions, were devised and used for the flow of water. However the

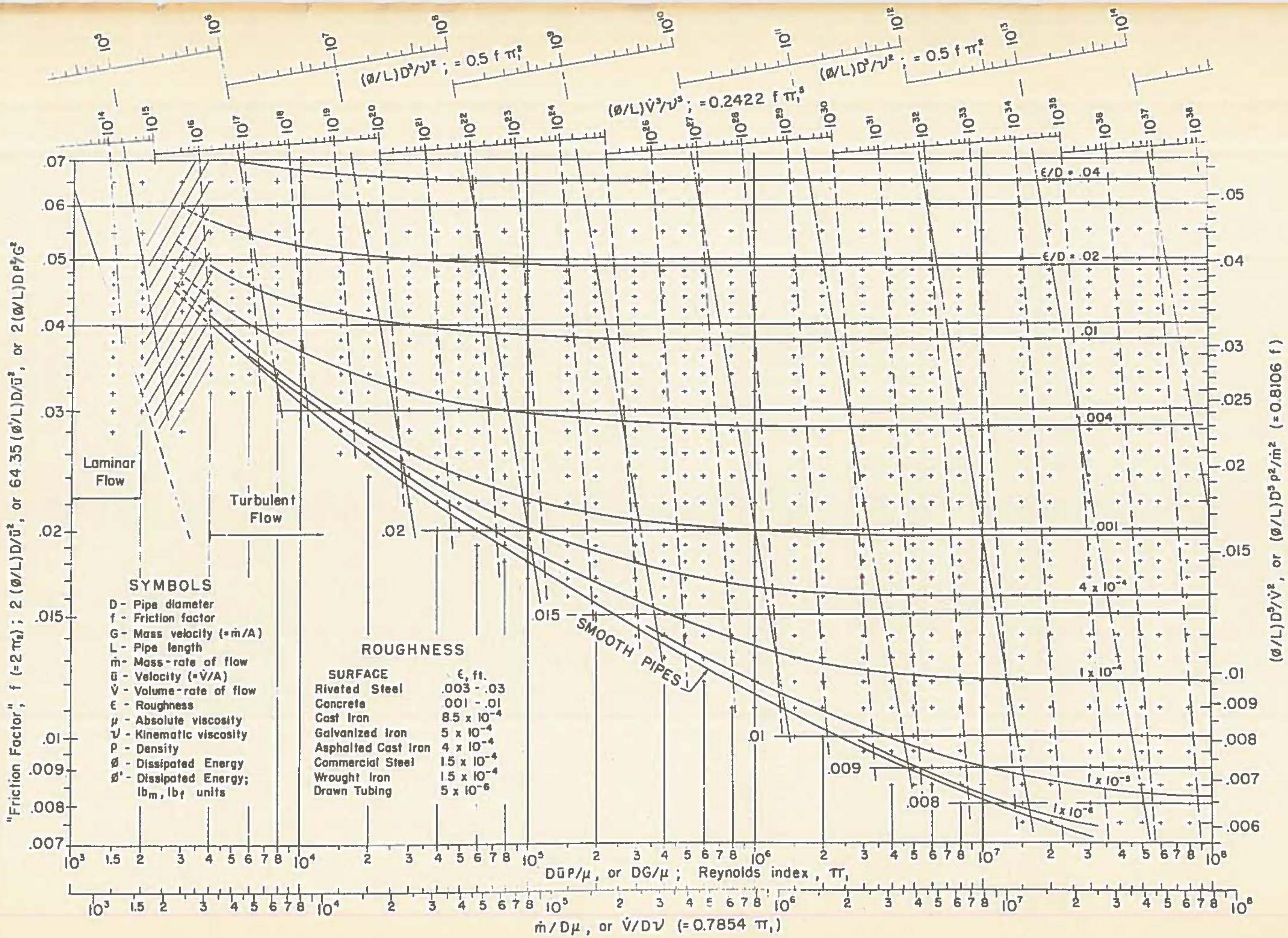

Fig. 6-3 MOODY DIAGRAM, SUPPLEMENTED

Fig. 6-3 exhibits the experimental findings, to logarithmic scales of the various dimensionless parameters. The major abscissa is that of the Reynolds (π_1, or N_R), but with a supplementary one in terms of mass or volume rates instead of velocity. The major ordinate scale is that of the "friction factor" (f), but with a supplementary scale again involving mass or volume rates instead of velocity. All scale magnitudes are ones conforming to the use of consistent units (such as the pound force, slug mass, foot and second), except as the friction factor was noted to be interpretable also in terms of ft-lbf/lbm.

The additional skewed coordinates and scales of the figure are those of the π_3 and π_4 of the last article, or $(\frac{\phi}{L})(\frac{\rho}{\mu})^2 D^3$ and $(\frac{\phi}{L})(\frac{\rho}{\mu})^5 \dot{V}^3$.

The curves of the figure are ones associating the various parameters (a) for the laminar flow range, (b) for the critical or unstable range, and (c) for predominantly turbulent flow. The straight line of the laminar-flow range is simple a graphical verification and representation, to logarithmic coordinates, of Eq. 6-5. Individual ones of the family of curves in the turbulent range represent the findings for pipes having relative wall-roughness as expressed numerically by the indicated ratio, ϵ/D.* Here ϵ (epsilon) denotes a representative mean dimension characterizing the size of the pits and excresoences at the wall surface, and the ratio expresses dimensionlessly their relative size. A generally reliable schedule of values of ϵ (in feet) appears in the figure, in relation to new pipes of various commerical classes. It would become unreliable for older and corroded or scale-encrusted pipe. Note that for a piping material of given "absolute" roughness, the relative roughness is inversely proportional to the pipe diameter.

*- Facilities for definitely associating relative-roughness indices with the Reynolds and friction-factor parameters, for flow regimes between very rough and very smooth pipe surfaces, were initiated by R.J.S. Pigott in the U.S.A. and C.F. Colebrook in England, and the organized findings of the latter were adapted to graphical representation in a diagram such as Fig. 6-3 by L.F. Moody in the U.S.A. In this country such a chart has become known as a Moody Diagram. See Mechanical Engineering, vol. 55, 1933, pp 497-501, 515; Journal of the Institution of Civil Engineers, London, vol. 11, 1938-39, pp. 133-156; and Transactions of the American Society of Mechanical Engineers, Nov. 1944, pp. 671-686.)

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term was absorbed in the constant, and different relative roughnesses were accomodated both by modifications of the constant and by using a value of n in connection with the D-term different from that with the u-term. Although the formulas were thus not dimensionally homogeneous and so invalid as general physical relations, by adroit manipulations they served their purposes with reasonable effectiveness.

Noteworthy features exhibited in Fig. 6-3 relate to the varying influences of the magnitude of the Reynolds index, at values greater than the critical, on the energy dissipation and friction factor. It is evident that

(a) for very rough pipes its influence disappears almost immediately; but

(b) for progressively smoother ones there is an initial resemblence between the lines for given ϵ/D and the laminar-flow line, but again an ultimate disappearance of its influence.

These observations may be regarded as evidence that

(a) full turbulence develops promptly <u>throughout</u> the cross-section of the very rough-walled pipe; but

(b) in smoother-walled ones the turbulence is initially limited to a core within a persisting annulus of laminar flow (i.e., a laminar boundary layer).

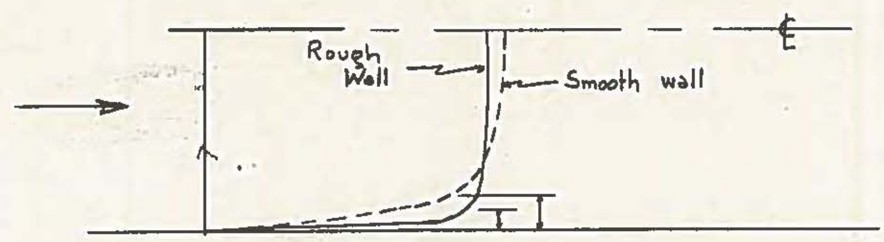


Figure 6-4

Figure 6-4 illustrates these situations. The relative thickness and persistence of this laminar layer so appears to be greater with the smoother walls, but to be much thinned and again be of minor influence at high values of the index.

Such considerations validate our earlier concept of the index as evidence of the dynamic character of flow. Also, the ultimate disappearance of its influence denotes that the inertial effects reflected in the numerator of the index (art. 6-3) have then become wholly dominant, in comparison with the viscosity effects reflected in its denominator.*

* - It is interesting that, in the idealized but essentially different flow of an <u>ideal</u> fluid (of zero viscosity and with no accompanying turbulence) <u>about</u> an obstruction which introduces directional changes and consequent inertial effects, variation of the index becomes again of no direct concern. But the index continues to be of paramount significance and utility in the <u>real</u> situations in which the effects of viscosity on boundary-layer formation and persistence is inescapable.

Following examples illustrate the use of Fig. 6-3 for several representative types of purposes.

Example 6-1. Estimate the pressure drop to be expected in a horizontal length of 500 feet of new galvanized iron pipe, of 0.82 inch internal diameter (3/4" nominal diameter) and through which fresh water at about 50°F is to be delivered at the rate of 15 gallons per minute.

Solution

Preliminary Items

$L = 500$ ft; $D = (.82/12 =)\ 0.0683$ ft; area $= 0.00367$ sq. ft.

$\dot{V} = 15/(7.48\times60) = 0.0334$ cfs; $\bar{u} = (.0334/.00367 =)\ 9.10$ ft/sec; $\bar{u}^2/2 = 41.4$ ft.lbf/slug, or 1.287 ft.lbf/lbm.

$\rho = 1.94$ slugs/cu.ft; $\dot{m} = (1.94 \times .0334 =)\ .0648$ slugs/sec; or $G = 17.7$ slugs/sec,ft^2.

$\mu =$ (fig.1-) 2.75×10^{-5} slugs/sec,ft; μ/ρ, or ν, $= 1.42\times 10^{-5}$ sq.ft/sec.

Reynolds index (π_1) $= (.0683\times9.10\times10^5/1.42 =)\ 4.4\times10^4$; and $\dot{m}/D\mu$, or $\dot{V}/D\nu$, $= .7854\ \pi_1 = 3.45\times10^4$.

Relative wall-roughness $= (.00005/.0683 =)\ 0.0007$.

Entering the abscissa of Fig. 6-3 with the above value of π_1 (or of $.7854\ \pi_1$), passing to intersection with the curve for ϵ/D of 0.0007, and thence to the ordinate scale, a friction factor of 0.024 is read. Thus

$$\phi = (024\times500\times9.10^2/2\times.0683 =)\ 7300 \text{ ft.lbf/slug},$$

$$\text{or } \phi' = \phi/32.17 = 227 \text{ ft.lbf/lbm.}$$

As $p_1 - p_2 = \rho\phi$ for a horizontal pipe, the pressure drop to be expected = 1.94×7300, or 62.4×227, $= 14{,}200$ lbf per sq. ft. and 99 psi.

Example 6-2. The pressure drop of 99 psi estimated in the above example is regarded as requiring excessive power cost for pumping. Estimate the size of like galvanized pipe which would enable the same rate of delivery but with a pressure drop reduced to about 30 psi.

Solution.

Preliminary Considerations

For the specified pressure drop, $\phi = (30\times144)/1.94 = 2230$ ft.lbf/slug, and $\phi/L = 4.46$ ft/sec^2. The other available data being those on $\dot{V}$, ρ and μ, the parameter $\frac{(\phi/L)\dot{V}^3}{\nu^5}$, or π_4 of Fig. 6-3, is established as $(4.45 \times \overline{.0334}^3 \times 10^{20}/1.42,$ or $4.45 \times 3.73 \times 10^{15}/5.77,) = 2.9 \times 10^{15}$.

Passing at this magnitude of π_4 to the ϵ/D line for about .0006 (see below), and thence down to the $\dot{V}/D\nu$ scale of the abscissa, that is established as 2.7×10^4. Thus

$$D = \left[(.0334 \times 10)/(2.7 \times 1.42) =\right] 0.087 \text{ ft, or } 1.05 \text{ inch.}$$

As a pipe of 1" nominal diameter has this internal diameter, the provision of this size of pipe will provide the desired decrease in pressure drop. Although this diameter corresponds to a value of ϵ/D of .00057, versus the above-used value of .0006, this is well within the precision with which values of actual relative roughness may be anticipated.

In the absence of the facilities provided by the π_4 scale of the figure, one might still proceed by a succession of approximations in which, for example, a number of diameters are assumed, and computations such as those of example 6-1 are made until that diameter is found for which ϕ and Δ p are as specified.

The very considerable reduction in pressure drop which the last example shows to have been produced by a quite moderate increase in pipe diameter may be surprising. But observe in Fig. 6-3 that, as at higher values of $\dot{V}/D\nu$ a large change in its magnitude causes a minor or perhaps negligible change in $(\phi/L)D^5/\dot{V}^2$ for a given character of pipe, the energy dissipation per unit length thus approaches <u>inverse</u> proportionality to the <u>fifth power</u> of the diameter when a given flow-rate is to be provided. Similarly, the energy dissipation in passage through a pipe of given diameter would approach <u>direct</u> proportionality to the <u>second power</u> of the required flow rate.

The accompanying table provides a schedule of convenient procedures in the use of the major and supplementary scales or coordinates of Fig. 6-4, when varied groups of (five) variables are known and a sixth is to be determined.

Table 6-3

Desired Item	Known or Assumed Data	Enter Graph by Scale for	Procedure
ϕ/L, expected	D, ρ, μ, and ϵ/D; and $\bar{u}$, $\dot{m}$, V or G	$D\bar{u}/\nu$ or DG/μ; or $\dot{m}/D\mu$ or $\dot{V}/D\nu$	Pass to line for ϵ/D and thence to scale for friction factor, $(\phi/L)D^5/\dot{V}^2$ or $(\phi/L)D^3/\nu^2$
D, required	$\phi/L, \rho, \mu$ and ϵ; and $\dot{V}$ (or $\dot{m}$)	$(\phi/L)\dot{V}^3/\nu^5$	Pass to selected ϵ/D line and thence to scale for $\dot{V}/D\nu$ or (or $\dot{m}/D\mu$), friction factor or $(\phi/L)D^5/\dot{V}^2$
$\dot{V}$(or $\dot{m}$), obtained	ϕ/L, D, ρ, μ and ϵ/D	$(\phi/L)D^3/\nu^2$	Pass to ϵ/D line and thence to scale for $\dot{V}/D\nu$ (or $\dot{m}/D\mu$)
μ, allowable	ϕ/L, D, ρ; ϵ/D and $\dot{V}$ (or $\dot{m}$)	$(\phi/L)D^5/\dot{V}^2$	Pass to ϵ/D line and thence to scale for $\dot{m}/D\mu$ or $\dot{V}/D\nu$
ϵ, allowable	ϕ/L, D, ρ; μ and $\dot{V}$ (or $\dot{m}$)	$\dot{V}/D\nu$, or $\dot{m}/D\mu$ and $(\phi/L)D^5/\dot{V}^2$	Pass to intersection of the two lines, and read the ϵ/D at that point

6-8 Localized Energy Dissipations. The valves, elbows and miscellaneous other fittings that are necessary accessories in any piping system introduce further energy dissipations which require individual consideration; although in some circumstances they are found to be relatively minor. Analytical predictions of their magnitudes are not practicable, and thus recourse is again had to the organization of test data on typical devices, through similarity considerations and dimensional analysis.

When undertaking such, note that the diameter/length ratio which appears in the friction factor for pipe flow $\left[\text{i.e., } f, = 2(\phi/\bar{u}^2)\ (D/L)\right]$ now becomes simply a dimensionless ratio and constant which would be ascribable to all of a series of geometrically similar fittings. A diameter alone, such as that of the pipe in which the fitting is installed, will suffice to describe the size of any individual representative. Employing volume or mass-rate as a rate index, rather than some mean velocity, the five variables ϕ, $\dot{V}$ (or $\dot{m}$),

D, ρ and μ thus remain to be collected into (two) dimensionless parameters.

Using the more direct π-method for their evolution, and considering first the four variables $\dot{V}$, D, ρ and μ.

$$(L^3T^{-1})(L)^a(ML^{-3})^b(ML^{-1}T^{-1})^c = f(M^0L^0T^0);$$

and for homogeneity in T, c = -1; in M, b = -c = 1; in L,

3 + a - 3b - c = 3 + a - e - 1 = 0, and a = -1. Thus

$$\pi_1 = \dot{V}\rho/D\mu \text{, or } \dot{V}/D\nu \text{ and } \dot{m}/D\mu .$$

Similarly for variables ϕ, $\dot{V}$ (or $\dot{m}$), D and ρ,

$$(L^2T^{-2})(L^3T^{-1})^a(L)^b(ML^{-3})^c = f(M^0L^0T^0);$$

and for homogeneity in M, c = 0; in T, a = -2; in L, 2 + 3a + b-3c =

2-6 + b = o, and b = 4. Thus

$$\pi_2 = \phi D^4/\dot{V}^2 \text{, or } \phi D^4\rho^2/\dot{m}^2 .$$

To summarize, for any of of a geometrically similar family of fittings or devices,

$$\phi D^4/\dot{V}^2 = \text{function of } (\dot{V}/D\nu) \text{ and } \lambda)$$

The parameter $\dot{V}/D\nu$ is recognized as the supplementary abscissa scale of Fig. 6-3. In a manner quite parallel to the negligible influence of $(\phi/L)D^5/\dot{V}^2$ which was seen to result from variation of this parameter with rough pipes of given ϵ/D, test findings indicate a like approach to constancy of $\phi D^4/\dot{V}^2$ in the typically disorganized and turbulent flow which occurs in conventional fittings and which persists for appreciable distances downstream there-from. The last relation thus reduces to

$\phi D^4/\dot{V}^2$ = a constant characterizing all of a series of geometrically similar fittings.

Table 6-3 quotes approximate magnitudes of this constant as found experimentally for various representative devices, of at least nominal geometric similarity. Also quoted are magnitudes of an alternative constant which is used extensively in engineering practice and is described as the "resistance factor" (k) for a type of fitting, is defined as $2\phi/\bar{u}^2$ or $64.35\phi'/\bar{u}^2$ and is equal to $(\phi D^4/\dot{V}^2)/0.811$.

Table 6-3

Valves

	$\frac{\phi D^4}{\dot{V}^2}$	$k_1 = \frac{2\phi}{\bar{u}^2}$
Globe, full open	8	10
Angle, full open	4	5
Swing-check, open	2	2.5
Gate, full open	.16	0.2
Fittings, screw-connecting		
Elbows, standard	0.7	0.9
" 2, medium sweep	0.6	0.75
" 2, long radius	0.5	0.6
", 45°	0.3	0.4
", 180°	1.8	2.2
Tee, via side outlet	1.5	1.8

Miscellaneous

	$\frac{\phi D^4}{\dot{V}^2}$	$k = \frac{2\phi}{\bar{u}^2}$
Abrupt diameter reduction		
10/1 ratio, at u = u	0.33	0.45
2/1 ratio, at u = u	0.27	0.33
Abrupt diameter increase		
10/1 ratio, at u = u	0.8	1
2/1 ratio, at u =	0.4	0.5
Entrance to square-cut		
Entrance to square-cut pipe end,		
if flush with wall	0.4	0.5
protruding from wall	0.8	1

A convenient practice when accounting the energy dissipation produced by a length L of pipe, plus one or another installed fitting, is to express that "equivalent" length of pipe (L') which would cause a dissipation equal to that of the combination. Or,

$$\phi_{aggregate} = .811(fL\dot{V}^2/D^5 + k\dot{V}^2/D^4)$$

$$= .811(f\dot{V}^2/D^5)\,(L + kD/f)$$

$$\text{or} \quad = .811f\,L'\,\dot{V}^2/D^5, \qquad (6\text{-}10)$$

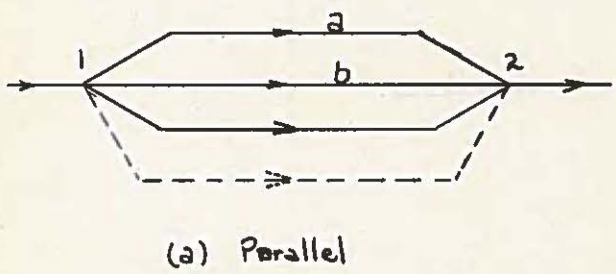
where L', = L equiv, = $L + k\,D/f$*.

6-9. Parallel and Series Piping Arrangements. Piping systems may become quite complex and the analysis of resulting "networks" rather troublesome, although generally manageable through the use of "relaxation" methods.** For present purposes it will be sufficient to note procedures for the analysis of simple parallel and series arrangements such as represented in sketches (a) and

* - Values of the ratio k/f have frequently been quoted as characteristic of given fittings, but it is evidently a function not only of the fitting but also of the character of the pipe.

** - Refer to Cross Hardy; Analysis of Flow in Networks of Conduits or Conductors, Univ. of Illinois Bulletin 286, Nov. 1946.

(b) of Fig. 6-4.



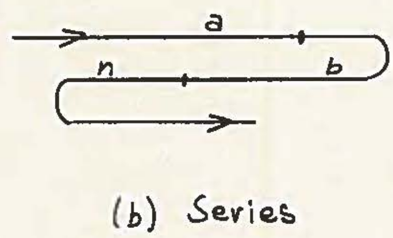


Figure 6-4

Distinguishing observations relating to each are that -

(a) For the parallel arrangement, (1) $\dot{V}$ total at sections 1 and 2 is the sum of the rates through the individual branches and (2) the pressure drop through each branch is the same, which leads to the conclusion that the specific energy dissipation (Ø) in each branch is also the same.

Forms of the steady-flow energy equation as adapted to these arrangements, and suggested methods of procedure with each, are indicated below. All are characteristically methods of progressive reduction of approximations which are necessitated by the multiplicity of variables. The material or relative surface roughness, length and internal diameter of each pipe section is presumed to be known, and symbol L' again denotes the equivalent length of pipe plus any installed fittings.

Parallel Arrangement. A suitable adaptation of the energy equation is =

$$
\begin{aligned}
{}_1\phi_2 &= g(z_1 - z_2) + (p_1 - p_2)/\rho + (u_1^2 - u_2^2)/2 \\
&= g(z_1 - z_2) + (p_1 - p_2)/\rho + .811\dot{V}^2(1/D_1^4 - 1/D_2^4); \\
\text{and } &= .811\, f_a L'_a\, \dot{V}_a^2/D_a^5, \\
\text{and } &= .811\, f_b L'_b\, \dot{V}_b^2/D_b^5, \\
\text{and } &= .811\, f_n L'_n\, \dot{V}_n^2/D_n^5.
\end{aligned}
$$

Situation A. To estimate the obtainable (total) flow rate, $\dot{V}$, and its distribution between the several branches, with an available and specified value of $g\Delta z + \frac{\Delta p}{\rho}$;

1) Initially neglecting $(u_1^2 - u_2^2)/2$, or thus regarding ${}_1\phi_2$ as approximately $\left(g\Delta z + \frac{\Delta p}{\rho}\right)$, enter Fig. 6-3 at each of correspondingly

computed magnitudes of $(\phi/L')D^3/\nu^2$ for each branch, pass to associated ϵ/D line for that branch, and thence to the $V/D\nu$ scale;

2) Compute trial values of the corresponding volume rate in each branch, and their sum;

3) Use this sum to evaluate the previously neglected difference in terminal kinetic energies, and repeat the above procedure, except now for entry at correspondingly modified magnitudes of $(\phi/L)D^3/\nu^2$.

Situation B. To determine the required $\left[g(z_1 - z_2') + (p_1 - p_2)/\rho\right]$ for enabling a specified aggregate flow rate $(\dot{V})$, and to ascertain its distribution through the several branches;

1) From ϵ/D information alone, adopt trial values of f and thereby of $\dot{V}$ for each branch, the latter by noting that

$$\left(\frac{\dot{V}_n}{\dot{V}}\right)^2 = \frac{D_n^5/f_n L_n'}{D_a^5/f_a L_a' + \cdots\cdots + D_n^5/f_n L_n'} ;$$

2) From corresponding trial values of $V/D\nu$ and the ϵ/D values, both for each branch, make revised selections of f and thereby of $\dot{V}$ for each branch;

3) From the aggregate of these compute $\left(g\Delta z + \frac{\Delta p}{\rho}\right)$.

Series Arrangement. The adaptation of the energy equation for this arrangement becomes -

$$_1\phi_2 = g(z_1 - z_2) + \frac{p_1 - p_2}{\rho} + .811\dot{V}^2\left(\frac{1}{D_1^4} - \frac{1}{D_2^4}\right),$$

and

$$= 0.811\dot{V}^2\left(\frac{f_a L_a'}{D_a^5} + \cdots\cdots + \frac{f_n L_n'}{D_n^5}\right).$$

Situation A. To estimate the obtainable delivery rate $(\dot{V})$, and the distribution of the energy dissipation, with a specified magnitude of $\left(g\Delta z + \frac{\Delta p}{\rho}\right)$

1) From the known ϵ/D data, select for each section trial values of f and thereby of ϕ_n, the latter by observing that $\phi_n/\phi =$

$$\frac{f_n L_n'/D_n^5}{f_a L_a'/D_a^5 + \cdots\cdots\cdots + f_n L_n'/D_n^5}$$, with ϕ taken tentatively as $\left(g\, dz + \frac{\Delta p}{\rho}\right)$;

2) Compute corresponding trial values of $\dot{V}\left[= \left(\frac{\phi_n D_n^5}{.811 f_n L_n'}\right)^{1/2}\right]$ and thereby of $\dot{V}/D\nu$ for each section;

3) Re-entering Fig. 6-3 with the ϵ/D data and these trial values of $V/D\nu$, make adjusted re-determinations of f and of ϕ_n/ϕ for each section, and of $\dot{V}$ for the series.

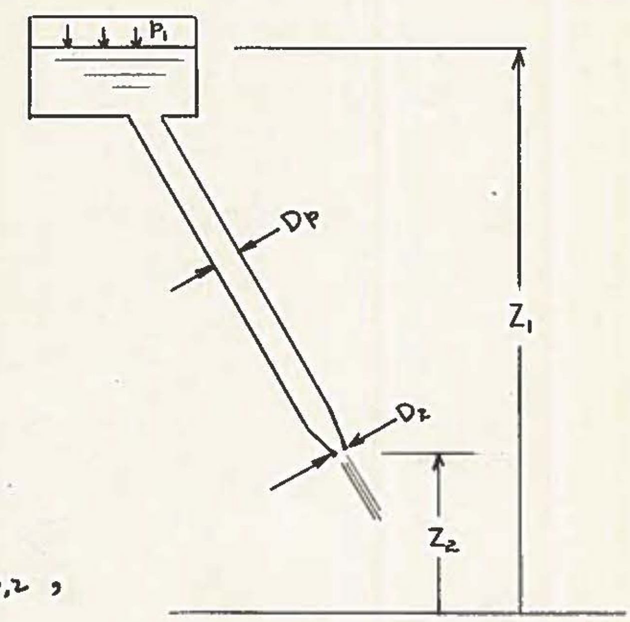
Situation B. To determine the required $\left(g\Delta z + \frac{\Delta p}{\rho}\right)$ for effecting a specified flow rate, and the distribution of the energy dissipation;

1) Enter Fig. 6-3 at individual values of $V/D\nu$, pass to appropriate lines of ϵ/D, and thence to scale of $\phi D^5/L'V^2$;

2) From these last determine ϕ for each section, their sum, and thereby the requisite $\left(g\,\Delta z + \frac{\Delta p}{\rho}\right)$.

In most situations a refinement of results by successive re-determinations is in principle to be desired. But it may by of uncertain benefit, because of the uncertainty with which the relative roughness of pipes or the length-equivalence of fittings may be appraised.

6-10. Discharge Coefficient, Pipe Line. It may on occasion be convenient to have an expression for the ratio between the actually obtainable rate of delivery from a tank or reservoir and through a given pipe line, and the rate ideally obtainable if there were no energy dissipation due to fluid friction (i.e., viscosity) and resulting turbulence. Fig. 6-5 represents such flow for a line of diameter D_p and friction factor f, from an elevated and possibly pressurized tank but one taken for convenience as of sufficient transverse area that the (downward) surface-velocity is negligible. With a nozzle of diameter D_2 at pipe end, the energy relation for the flow at any instant is closely,



$$gz_1 + \frac{p_1}{\rho} = gz_2 + \frac{p_2}{\rho} + \frac{\bar{u}^2}{2} + {}_1\phi_2$$

Or $$g(z_1 - z_2) + \frac{p_1 - p_2}{\rho} = e_{zp,1} - e_{zp,2},$$

$$= \left(\frac{\dot{V}}{A_2}\right)^2\Big/2 + .811\,\frac{fL'\dot{V}^2}{D_p^5}$$

$$= .811\,\dot{V}^2\left(\frac{1}{D_2^4} + \frac{fL'}{D_p^5}\right)$$

and $$\dot{V}_{actual} = 1.11\,D_2^2\left[\frac{e_{zp,1} - e_{zp,2}}{1 + fL'D_2^4/D_p^5}\right]^{1/2} \qquad (6\text{-}10a)$$

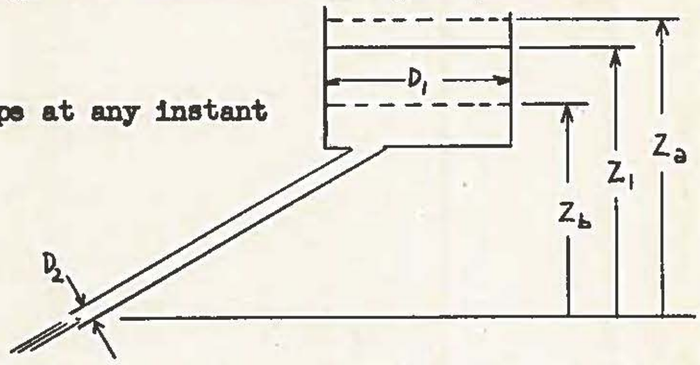
For idealized conditions, at ${}_1\phi_2$ = zero,

$$\dot{V}_{ideal} = 1.11\,D_2^2\,(e_{zp,1} - e_{zp,2})^{1/2}.$$

The flow-rate ratio, expressing a discharge coefficient (C_{disch}) for the system, so becomes

$$C_{disch} = \frac{\dot{V}_{actual}}{\dot{V}_{ideal}} = \left[1 + fL'\left(\frac{D_2^4}{D_p^5}\right)\right]^{-1/2}. \qquad (6\text{-}10b)$$

6-11. Emptying and Filling. The foregoing analyses have related solely to steady-flow conditions. Although the general case of unsteady flow may lead to rather complex analysis, the special case of the emptying or filling of a vessel by gravity or with constant pressure assistance is one which is relatively simple, yet of frequent concern. Fig. 6-6 illustrates an open cylindrical vessel being gravity drained.



The velocity of the water in the pipe at any instant will be a function of the height of liquid still remaining in the tank. However, the velocity in the vessel is simply

$u_1 = dz/dt$

and the time required for the level to change by an amount Δz is

$$\Delta t = \int_{z}^{z+\Delta z} dz/u_1 \qquad (6\text{-}11)$$

From the energy equation

$$\frac{u_1^2}{2} + gz_1 = \frac{u_2^2}{2} + {}_1\Phi_2$$

Assuming that all of the frictional loss occurs in the pipe,

$${}_1\Phi_2 = \frac{fL}{D_2}\frac{u_2^2}{2} = \frac{fLu_1^2}{2D_2}\left(\frac{D_1}{D_2}\right)^4$$

Hence

$$u_1 = \sqrt{2gz_1\left[\left(\frac{D_1}{D_2}\right)^4 + \frac{fL}{D_2}\left(\frac{D_1}{D_2}\right)^4 - 1\right]} = \frac{(D_2/D_1)^2\sqrt{2gz_1}}{\sqrt{1 + \frac{fL}{D_2} - \left(\frac{D_2}{D_1}\right)^4}} \qquad (6\text{-}11a)$$

The term $\sqrt{\dfrac{1}{1+\frac{fL}{D_2}-\left(\frac{D_2}{D_1}\right)^4}}$, except for possible variation in the friction factor, f is a constant and can be designated a flow coefficinet, C_f, for the system. If the variation of C_f is minor and negligible equation 6-11 integrates to

$$\Delta t = \frac{(D_1/D_2)^2}{C_f\sqrt{2g}}\int_{z_a}^{z_b}\frac{dz}{z^{1/2}} = \frac{A_1/A_2}{C_f\sqrt{2g}}\left(\sqrt{z_a} - \sqrt{z_b}\right) \qquad (6\text{-}11b)$$

If the tank diameter, D_1, varies with z, or the variation of f with respect to z is appreciable, then the above integration may be done graphically. In the event that the vessel is closed and there is an applied pressure, p, to the surface of the liquid then the energy equation is modified to include this item. Similarly, many other variations of the problem are possible, with corresponding solutions following the same pattern, but with somewhat greater complexity.

6-12. Flow of Gases Through Pipes. For the flow of expansible fluids at very moderate flow-densities, and through shorter pipes, the pressure and consequent density and velocity change en route may be sufficiently small that use of the preceding relations (for liquid flow) may still be suitable. But at greater flow rates, or through longer pipes, the decreasing gas density and increasing stream velocity, resulting from progressive pressure drop, will require more adequate methods of accounting.

Thermodynamics analyses beyond the scope of this material are necessary for enabling justification of such methods. For those the reader is referred to current thermodynamic literature.* However, resulting relations which are adequate for many situations are quoted herewith. Also a more informative graphical representation is provided, suitable for the flow of diatomic gases (including air) and indicating further a unique feature encountered in extreme situations.

These relations are -

$$p_1^2 - p_2^2 = G^2 R \bar{T} \left[\frac{fL}{D} + \ln\left(\frac{p_1}{p_2}\right)^2 - \ln\left(\frac{T_1}{T_2}\right)^2 \right]; \qquad (6\text{-}12a)$$

or if the possibly minor logarithmic terms are neglected,

$$p_1^2 - p_2^2 = (fL/D)\, G^2 R \bar{T} \qquad (6\text{-}12b)$$

The last relation may be put in the equivalent form

$$p_1 - p_2 = (fL/D)(G^2/2\rho) \qquad (6\text{-}12c)$$

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* - Not surprisingly, the material of chapter 16, Engineering Thermodynamics, Kiefer, Kinney and Stuart, John Wiley and Sons, coordinates well with the following.

In these

D = inside diameter, feet   p = abs. pressure, lbf/ft^2

f = friction factor, Fig. 6-3   R = gas constant, ft.lbf/(slug, ^oR),

G = flow density, slugs/(sec,ft^2)   and (154x32.17/28.97 =) 1716

L = pipe length (equiv.), ft   for air (art. 1-7)

$\bar{T}$ = a representative mean temperature of the gas while en route through the pipe, ^oR (= ^oF + 460)

= representative mean density of gas, $\bar{p}/R\bar{T}$, slugs/cu.ft.

Figure 6-6 associates graphically the variables involved in Eg. 6-12a, but for the specific situation of the effectively adiabatic flow of diatomic gases at moderate pressure and temperature levels. For such representation it is advantageous to have employed as coordinates

(a) the ratio p/p_o between the local pressure, at a given point along the pipe, and the pressure in a storage reservoir from which the gas is presumed to be passing to the pipe through a smoothly convergent entry fitting; and

(b) the ratio G/G^* between the actual flow density and a uniquely maximum rate (G^*) at which the gas might ideally issue from the reservoir through the fitting alone, with no appended pipe (i.e., zero length of pipe.

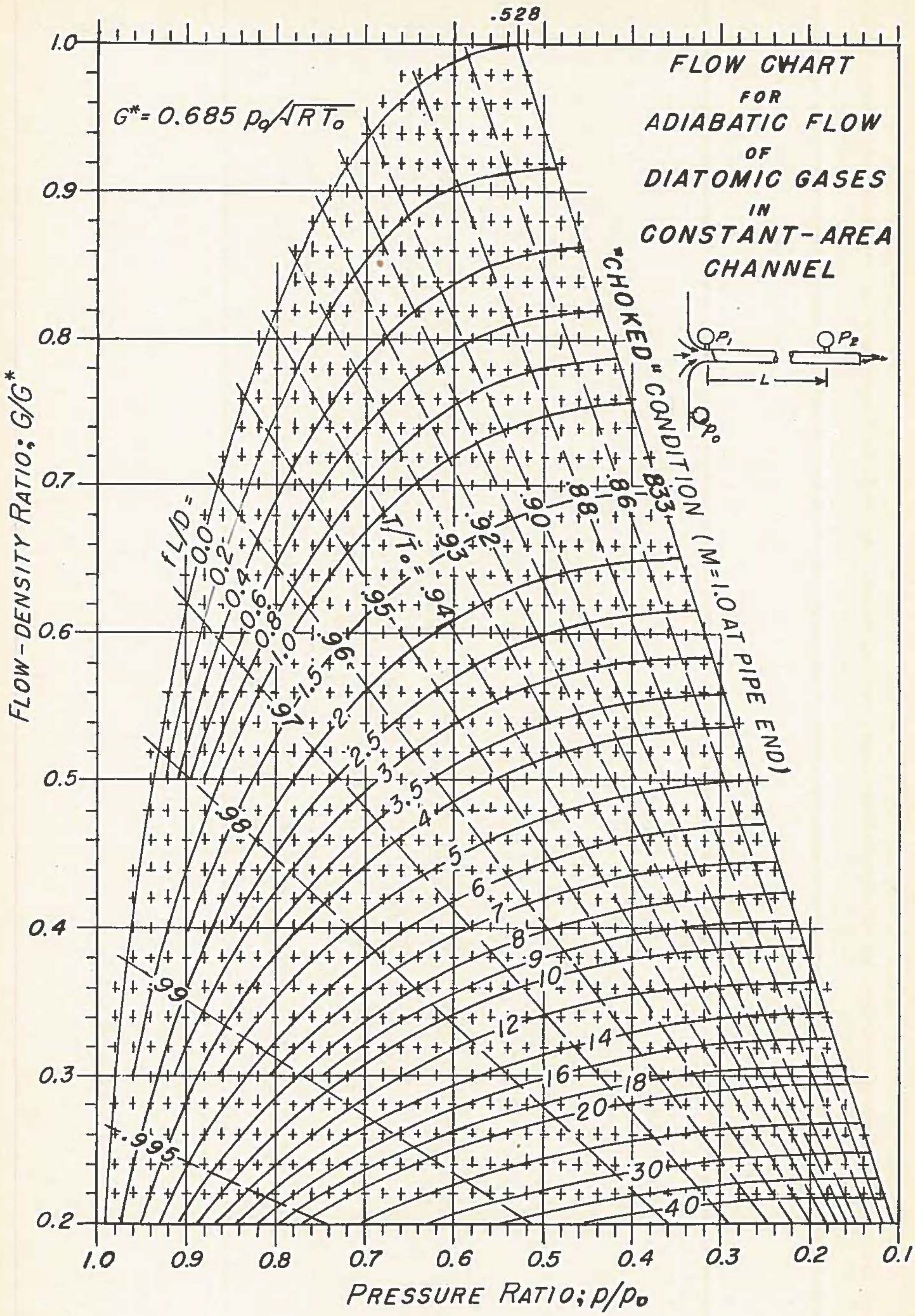
Thermodynamic analysis establishes that, again for diatomic gases at moderate pressure and temperature levels,

$$G^* = 0.685\ p_o/\sqrt{R\ T_o} \tag{6-13}$$

The curved contour lines of the figure are those of fL/D. Their termination, to the right, denotes the significant situations that

(a) There is a limiting or maximum flow density which can be obtained in flow through a pipe of given fL/D. It is that corresponding to the value of G/G^* established by the terminus of the curve, this establishing also the accompanying relative pressure (p/p_o) to which the gas must have been permitted to expand in passage to the exit end of the pipe. Also -

(b) There is a limiting or maximum length of pipe of given D and f in which a specified relative flow density G/G^* may be obtained. It is that corresponding to the fL/D line which co-terminates with the G/G^* line, this terminus establishing also the minimum pressure to which the gas might and



FLOW CHART FOR ADIABATIC FLOW OF DIATOMIC GASES IN CONSTANT-AREA CHANNEL
$G^* = 0.685\, p_0/\sqrt{RT_0}$
"CHOKED" CONDITION (M=1.0 AT PIPE END)
.528
fL/D =
T/T_0 =
FLOW-DENSITY RATIO; G/G^*
PRESSURE RATIO; p/p_0
p_1
p_2
p_0
L

Fig. 6-6

also must have expanded in passage to the exit end of the pipe.

The lines radiating from the lower right are contours for constant magnitudes of the temperature ratio T/T_o. For the adiabatic flow of diatomic gases at moderate pressures and temperatures the magnitude of this ratio for the line joining termini of all fL/D lines is 0.833.

The stream velocity attained on accelerative expansion of the gas to this minimum terminal temperature, in passage to the exit end of the pipe, is uniquely equal to the velocity with which weak compression-and-rarification waves will advance through a gas when at this temperature. This velocity is known also as a <u>sonic</u> velocity, as sound waves of moderate intensity are of this character. It is in general a function of the elasticity characteristics $\left(\frac{\partial \rho}{\partial p}\right)_s$ of the gas.

The ratio of the speed of advance of a stream of gas to the sonic velocity, both at the same temperature, is known as the <u>Mach number</u> characterizing the stream. This dimensionless ratio parallels the Reynolds index in utility, in serving as an index of the influences of its compressibility in modifying the character of the flow of a compressible fluid. For similarity of flow pattern with such a fluid, in passage through or about geometrically similar devices and in circumstances where inertial effects, viscosity <u>and</u> elasticity are mutually operative in establishing the character of the flow, it would in principle be necessary that particular magnitudes of <u>both</u> parameters be maintained constant.

The items procurable through use of Fig. 6-6 are perhaps better illustrated by the following example.

<u>Example 6-3</u>. For the flow of air through a horizontal duct, or relative roughness and at a flow rate such that f is constant at about 0.03, determine the following items through the use of Fig. 6-6, and also make comparisons as indicated below.

The air is supplied to the duct through a smoothly convergent entry fitting from a large reservoir in which it is maintained at 114.7 psia and 100°F.

(a) Determine the pressure and temperature at entry end of duct, and the rate of flow obtained, if a pressure of 95.2 psia is maintained at the exit end and the duct is of 100 feet length.

Also compare the flow rate as so determined with that as computed through the use of equations 6-12 .

(b) Determine the maximum length of pipe through which the above flow rate could be maintained, and the air pressure (and temperature) at exit end of duct, if this or a less pressure exists in the region which the duct is delivering.

(c) Determine the maximum flow rate obtainable through the specified pipe, and the associated pressure at exit end, if that or a less pressure is maintained in the delivery region.

Solution. $A = (\pi/4)(6/12)^2 = 0.196$ sq.ft; $G^* = 0.685\text{x}(114.7\text{x}144)/\sqrt{1716\text{x}560}$ $= 11.55$ slugs/(sec.ft^2); $f/D = .03/.5 = .06$ ft^{-1}.

(a) At $fL/D = (.06\text{x}100 =)\ 6.0$, and $p_2/p_o = (95.2/114.7 =)\ 0.85$, $G/G^* = 0.3$ (by Fig. 6-6), $G = (.3\text{x}11.55 =) = 3.47$ slugs/(sec,ft^2) or 111.5 lbm/(sec,ft^2), and $\dot{m} = (111.5\text{x}.196 =)$ 21.9 lbm/sec. Also, $p_1 = (.98\text{x}114.7 =)$ 112.4 psia, $T_1 =$ $(.994\text{x}560 =)\ 557^\circ R$, and $T_2 = (.991\text{x}560 =)\ 555^\circ R$.

Using Equation 6-12a

$$G^2 = \left[144^2(112.4^2 - 95.2^2)\right] / (1716\text{x}558)\left[6.0 + \ln(112.4/95.2)^2 - \ln(557/555)^2\right]$$
$$= (7.40\text{x}10) / \left[(9.58\text{x}10)(6.0 + 0.33 - .007)\right]$$
$$= 77.2/6.337 = 12.2;\ \text{and}$$

$G = 3.49$ slugs/(sec, sq ft), which is in close agreement with the above 3.47 slugs/(sec, sq ft).

Using equation 6-12b , or neglecting the logarthmic terms,

$G = \sqrt{77.2/6.0} = 3.59$, in comparison with above 3.49.

To use equation 6-12c , and employing an algebraic mean density of $\left[2(144/1716)(112.4/557 + 95.2/555) =\right]$ 0.0156 slugs/cu ft,

$G^2 = 144(112.4-95.2)(2\text{x}0.0156/6.0) = 12.9$; and

$G = 3.59$ slugs/(sec, sq ft)

It is evident that employment of the simpler relations may introduce appreciable error, that the use of any of these relations for pressure determinations rather than ones for flow-density or duct-length would require procedure through successive approximations, and also that none even intimate the limitations associated with approach to or attainment of sonic velocity in the duct.

(b) At the terminus of the line for $G/G^* = 0.3$, $fL/D = 19$ and $p/p_o = 0/16$. These signify that the <u>maximum pipe length</u> through which the above flow density of 3.47 slugs/(sec, sq ft), or flow rate of 21.9 lbm/sec, could be obtained is ($19x.5/.03 =$) 317 ft; and that for obtaining it a discharge-region pressure not exceeding ($0.16 \times 114.7 =$) 18.35 psia would be required.

(c) At the terminus of the line for $fL/D = 6.0$, $G/G^* = 0.47$ and $p/p_o = 0.25$. These signify that the <u>maximum flow density</u> obtainable through a line of the specified length and character is ($11.55 \times .47 =$) 5.43 slugs/(sec, sq ft), or $\dot{m}$ max $= 34.2$ lbm/sec, and that for obtaining it a discharge-regions pressure not exceeding ($0.25 \times 114.7 =$) 28.7 psia would be required.

<u>Problems 6-14.</u>

1. Determine the following items relating to a 6 inch (diameter) pipe in which there is laminar type of flow of oil at the rate of 300 gpm, the oil having a specific gravity of 0.93 and a viscosity of 45 seconds Saybolt Furol:-

(a) Magnitude of the Reynolds index corresponding to the specified flow conditions; (1710)

(b) Pressure drop per 1000 foot length of pipe, and the longitudinal force required to retain the pipe in position; (5.3 psi, 153 lb)

(c) Energy dissipation per pound of oil passing, per foot length of pipe, and the power correspondingly required by pumps located at 25 mile separation for actuating the flow; (.0136 ft.lb/lb, 126 hp)

(d) Ratio of center-line to mean velocity.

2. At what velocity should air be caused to approach a 1/2-size model of an airplane if flow conditions about the model are desired that are dynamically similar to those about the full-size plane when travelling at an air speed of 80 miles per hour, conditions of the air being the same in both cases? If a much smaller model were used, requiring excessively high air speeds, what alternative might be employed for avoiding such speeds?

3. For two geometrically similar fittings (for piping) with internal diameters at corresponding sections of 2" and 10", at what velocity should water be caused to pass through the first in order that the flow conditions shall be dynamically similar to those of water at the same temperature flowing at 4 ft/sec through the second?

4. Gasoline of a viscosity of 0.5 centipoises and 0.716 specific gravity at 50^oF is being pumped through a 4" hose at the rate of 450 barrels per hour

(42 gal/bbl). Compute the magnitude of the Reynolds index (in consistent units) that will designate the dynamical condition of the flow. (356,000)

5. If in flow through an open channel wave-making effects contribute to the character of the flow pattern, and are influenced by the local force of gravity per unit mass (i.e., g), associate by use of the π- method the variables size, velocity, density and g in a dimensionless parameter (or the "Froude" index) that will serve as an index of the character of the wave-motion component of the fluid flow.

6. Check the results of problem 1 (c) by the use of Fig. VI-3.

7. Check the results of example 6-1 (art. 6-6) by taking the parameter $\left(\frac{\Phi}{L}\right)\frac{D^5}{\dot{V}^2}$ from fig. 6-3.

8. Energy-dissipation data is to be organized for a portion of a pipe in which disturbances of some nature are causing the flow pattern to change along the pipe. A result is that $\emptyset$ is not necessarily proportional to L, and that a modified character of functional relation may be necessary. For this situation provide suitable dimensionless parameters by collecting (a) terms $\emptyset$, $\dot{V}$, D and ρ; (b) $\dot{V}$, D, ρ and μ; and (c) D, ρ, μ and L. Discuss briefly the significance of π_c, as so evolved, as a "shape" factor.

9. Check the results of example 6-1 (art. 6-6) by obtaining values of the parameters $(\Phi/L)\,D^3/\nu^2$ and $(\Phi/L)\,\dot{V}^3/\nu^5$ from Fig. 6-3.

10. For the 1" pipe selected in example 6-2 (art. 6-7) estimate the attainable delivery rate with the specified pressure drop of 30 psi if by fouling with age the line was reduced 10% in effective diameter and also transferred to the "fairly rough" category.

11. Oil is to be pumped at the rate of 350 bbl (42. gal/bbl) per hour through a 3" fairly smooth line of an equivalent length of 1000 feet, rising 75 feet in this distance. The oil is at 35°F, has a specific gravity of 0.925 and its kinematic viscosity in sq.cm/sec is expressible by the relation = $4200/(t, {}^{\circ}F)^{2.05}$.

(a) What pressure will be required at the pump discharge if the pump is at the reservoir at the lower level, and will an available 35 hp motor (maximum capacity) be capable of driving the pump if the pump efficiency is about 0.60%?

Ans. 246 psi; No

(b) If this motor has to be used, to about what temperatures would the oil need to be heated (for viscosity reduction) in order to enable the desired pumping rate if the flow continues laminar, and to what temperature if pulsations induced turbulent flow? Ans. 47°, 102°

12. For draining an open tank to a sump at 500 feet distance, and 16 feet below the surface of the water in the tank, a lot of badly corroded 3" pipe is available. A run-off of at least 100 gpm is desired. With the water at about 68°F, will this pipe probably serve?

13. By ascertaining the approximate slope of the line for fairly smooth pipe (Fig. 6-3) in the vicinity of the point corresponding to the conditions of example 6-1 (art. 6-7) evolve an equation by which the value of ϕ might have been expressed. Also write the equation in a reduced form suitable only for water at about 68°F (i.e., with the viscosity term included in the constant of the equation). Check your equations by use for recomputations of ϕ . Also compare your equations with any that may be found in an available handbook.

14. For the pipe and flow of example 6-1 (art. 6-7) estimate the supplementary energy dissipation for the further condition that inflow to the line is from a reservoir into which the pipe protrudes, the pipe discharges slightly below the surface of a second reservoir, and the line contains 6 short-radius 90° elbows and 2 (open) globe valves. Also express the total equivalent length of the pipe with fittings et cetera.

15. A 6" manifold supplies one very smooth 3" hose 100 feet long, and second fairly smooth 2 1/2" hose 300 feet long. Both discharge to the atmosphere through nozzles giving 1" jets, at about the same level as the manifold. The pressure at the manifold is 85 psig. Estimate the rate of flow through each hose, and the aggregate. Ans. 456 gpm total

16. A 125 feet length of new 1" smooth pipe, with open globe valve, is used to extend an existing badly corroded 2" pipe 400 feet long. The entire line is about horizontal. About what rate of discharge to the atmosphere may be obtained if the supply pressure is 50 psig, and about what will be the pressure at the junction of the two pieces?

17. A vertical tank of water of 10 feet diameter is to be emptied through a 6" fairly rough pipe with an equivalent length of 175 feet. The base of

the tank is 25 feet above the open end of the pipe, and the water is initially 30 feet deep in the tank. Estimate the time to empty the tank by gravity flow, and also the required (air) pressure at the surface of the water in the tank in order to halve the emptying time. Ans. 17.3 min

18. Find the required air pressure at the surface of the water in the tank of problem 17 in order to halve the emptying time.

CHAPTER 7

FLOW METERING AND CONTROL

7-1. Foreword. In his practice it is frequently necessary that the engineer measure and/or control the rate of fluid flow to or from his equipment. For such purposes information on a mean rate may be sufficient, and may be secured by simultaneous quantity and time determinations, with the first being made in successive "batches" and perhaps using a positive-displacement type of meter.

Or information on the rate of flow at any instant may be more essential, with determination of the total quantity in a given time interval being made by a rate-time integration. The following discussions relate in the main to the devices customarily employed for such rate determinations in confined streams.

For all of these the interpretation of the evidence which they provide is based rather directly on the steady-flow energy equation, but as expressed in terms of flow rates rather than in direct terms of velocities. That is, to make such an adaptation of the energy (and continuity) equation, for incompressible fluid flow,

$$gz_o + \frac{p_o}{\rho} + \frac{\bar{u}_o^2}{2} = gz + \frac{p}{\rho} + \frac{\bar{u}^2}{2} + \phi$$

and $$\frac{\bar{u}^2 - \bar{u}_o^2}{2} = g(z - z_o) + \frac{p_o - p}{\rho} - \phi$$

or, to abbreviate, $$= \left[(e_{zp})_o - (e_{zp})\right] - \phi$$

where subscript $_o$ denotes items relating to some convenient reference position along a stream, and items without subscript refer to any other significant position, except that as the symbol ϕ denotes the energy dissipation in passage between the two positions. But if the velocities are those of streams of known (actual or nominal) transverse areas, by introduction of the continuity equation ($\dot{V} = \bar{u}\,A$)

$$\dot{V} = \sqrt{\frac{2\left[(e_{zp})_o - (e_{zp}) - \phi\right]}{1/A^2 - 1/A_o^2}} \qquad (7\text{-}1)$$

or, in terms of a flow density $\left(G, = \dot{m}\rho \text{ or } \frac{\dot{m}}{A}\right)$

$$G = \rho\sqrt{2\left[(e_{zp})_o - (e_{zp}) - \phi\right.} \qquad (7\text{-}1a)$$

where further $G = \dot{m}/A$ or $G_o = \dot{m}/A_o$

Several representative devices which are employed, and details of further energy-equation modifications are as follows:

7-2. Venturi-meter. As is seen in Fig. 7-1, the essential parts of the venturi meter are a smoothly converging cone, through which the liquid enters from a pipe line in which the instrument is installed, and means for transmittal of the local pressures at entrance and exit sections of the cone to equipment which measures their pressure-difference. The functions of the diverging cone, which is known as a diffuser and connects the throat of the convergent cone with the pipe through which the stream departs, are the continued confining and control of the stream and the provision of favorable facilities for its deceleration to the entry velocity with minimum energy dissipation. This enables an almost complete regain of the entry pressure.

The volumetric rate of flow of the liquid is evidently expressible as

$$\dot{V} = A_2\sqrt{\frac{2[(e_{zp})_1 - (e_{zp})_2 - \phi]}{1 - A_2^2/A_1^2}}$$

where area A_1 and A_2 are those at, respectively, the entry and throat sections of the convergent cone. Also

$$\dot{V} = G_1 A_1/\rho \quad \text{or} \quad G_2 A_2/\rho$$

For utilization of the arrangement as a meter, direct or indirect accounting of the effects of the energy dissipation (ϕ), must evidently be made. In the venturi meter this is evidenced by an actual flow rate less than that which might ideally be anticipated from a simultaneously measured magnitude of $(e_{zp})_1 - (e_{zp})_2$. Accounting for the energy dissipation is made indirectly but validly in terms of the (dimensionless) ratio $\dot{V}_{actual}/\dot{V}_{ideal}$, which is known as a discharge coefficient (C_d) and for a given meter is determined experimentally by independent measurements of actual flow rates and concurrent observations of $(e_{zp})_1 - (e_{zp})_2$. With data available, from any source, concerning the value of the coefficient,

$$\begin{aligned}\dot{V} &= C_d \dot{V}_{ideal}\\ &= C_d A_2\sqrt{\frac{2[(e_{zp})_1 - (e_{zp})_2]}{1-(A_2/A_1)^2}}\\ \text{or } &= C_d A_1\sqrt{\frac{2[(e_{zp})_1 - (e_{zp})_2]}{(A_1/A_2)^2 - 1}}\end{aligned} \tag{7-2}$$

and, in terms of flow densities,

$$G = C_d \rho\sqrt{\frac{2[(e_{zp})_1 - (e_{zp})_2]}{(A_1/A_2)^2 - 1}} \tag{7-2a}$$

where G is that either at entry ($= \dot{m}/A_1$) or at throat ($= \dot{m}/A_2$).

As was the case for the flow through pipes or fittings, for venturi tubes of geometrically similar form and area-proportions the magnitude of ϕ, and thus in turn that of C_d, is found to be again a function of the Reynolds index, except as moderately influenced by the relative surface roughness of tubes of different sizes.

The Reynolds index may be expressed with somewhat greater convenience in its alternative form DG/μ. The diameter and flow density are those for either the entry or the throat section of the meter, but also with D_2G_2/D_1G_1 being seen to equal D_1/D_2. The typical character of the empirical relation between C_d and the Reynolds index is indicated also in Fig. 7-1*. In the figure, abscissa scales of DG_{ideal}/μ and DG_{actual}/μ are both shown, the latter computed directly from calibration test data. The DG_{ideal}/μ scale is constructed by simple dividing DG_{actual}/μ by the corresponding value of C_d.

With larger tubes of low relative roughness, and operating at higher Reynolds index, values of C_d slightly exceeding 1.0 may in fact be encountered, but are not to be interpreted as evidence of a less-than-zero energy dissipation. They are instead a consequence and suitable compensation for the generally moderate impropriety (art. 5-6) of kinetic-energy evaluations as $\bar{u}^2/2$.

Precautions taken in the installation and use of the meter include a location sufficiently downstream (10 to 20 diameters) from any fitting disturbing the flow, and maintenance of a throat pressure sufficiently high to avoid cavitation.

7-3. Flow Nozzle, and Orifice. In a sense the flow nozzle and the orifice are merely simplifications of the venturi tube. Both provide for the production of an initially convergent and accelerating stream, and for inferential determination of the flow rate by measurement of an associated pressure change, but they lack the diffuser section of the venturi tube. Other departures in their form, and observations concerning the interpretation of the pressure data, are considered individually in the following.

(a) Flow Nozzle. Fig. 7-2 indicates a typical flow nozzle, with provisions made to obtain access for pressure-difference measurements between upstream and downstream sides of the nozzle.

A quite reasonable query, relative to the latter, is whether the positions of the access "slits" are such that the pressures there will be truly repre-

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* - Values of the coefficients are as reported in "Fluid Meters, Their Theory and Application", Amer. Soc. of Mech Eng., 4th Ed., 1932.)

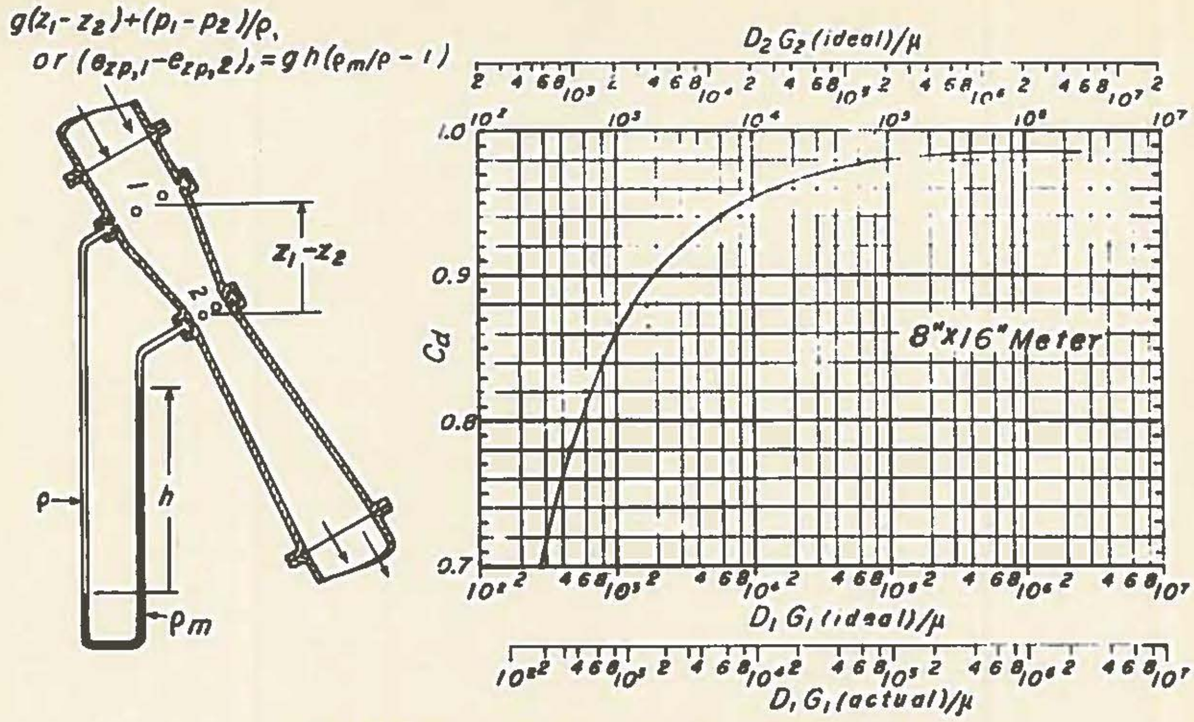

Fig. 7-1. Venturi meter, and Calibration curve

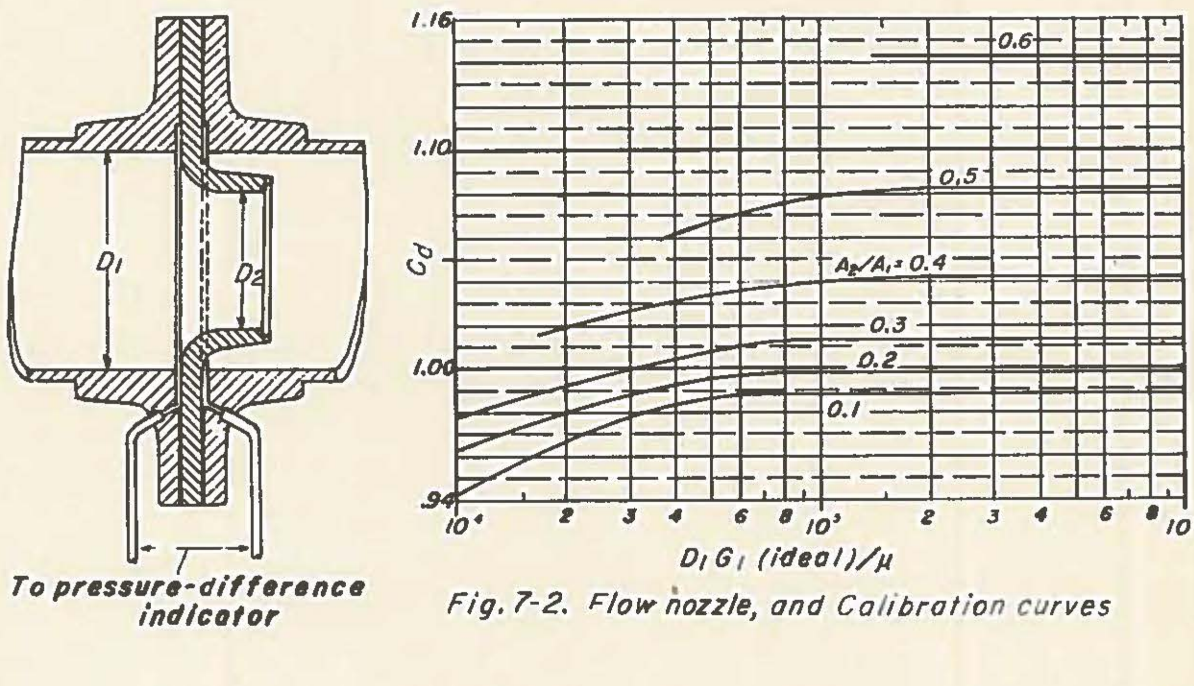

Fig. 7-2. Flow nozzle, and Calibration curves

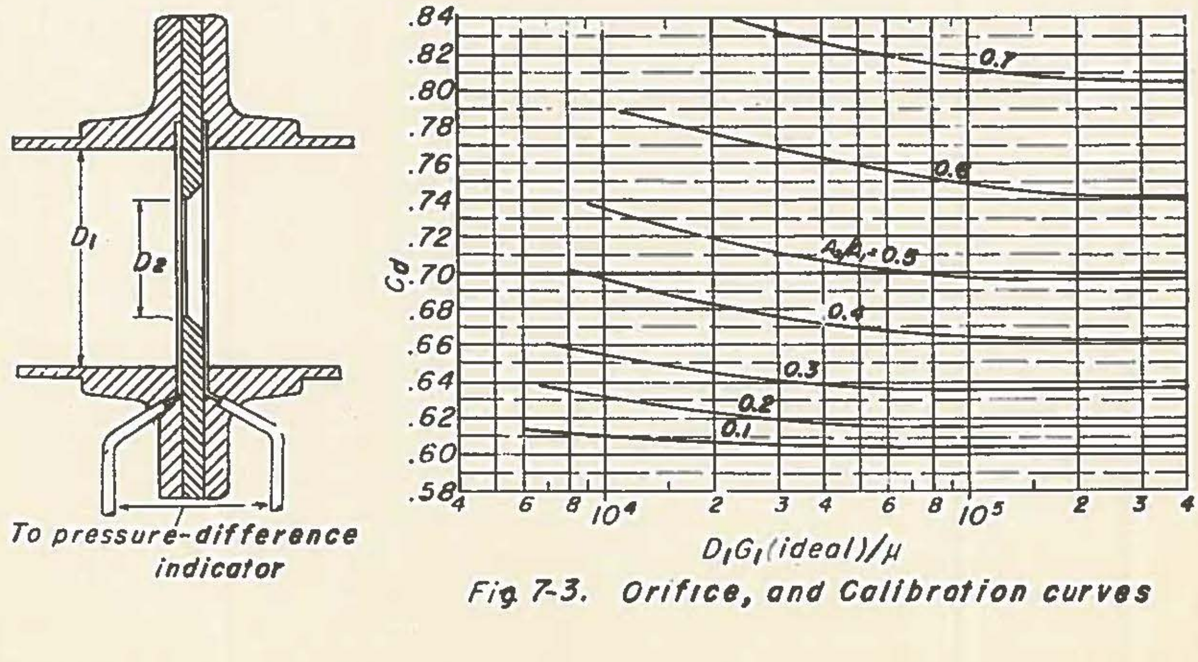

Fig. 7-3. Orifice, and Calibration curves

sentative of those in the approaching stream of area A_1, or in the stream of area A_2 departing through the nozzle throat. Actually they are not, and there might thus appear to be little justification for associating such incompatible pressures and areas in equations such as 7-2 or 7-2a.

However there is ample experimental verification that

(1) dynamic similarity of flow pattern exists both in and <u>in the vicinity</u> of geometrically similar nozzle arrangements if the magnitudes of the Reynolds index are the same and that therefore

(2) even such pressure determinations will still serve to provide adequate evidence of flow similarity.

Thus utilization of the above relations becomes in fact suitable, but it is not to be anticipated that actual flow rates will be related to (nominally) ideal ones by discharge coefficients which agree with ones such as indicated for the venturi tube in Fig. 7-1.

The curves of Fig. 7-2 depict typical magnitudes and the manner of variation of such discharge coefficients, for flow through nozzles of the indicated form and a variety of area proportions.* Here the appearance of coefficients greater than 1.0 is due to the fact that the pressure differences actually measured (and on which computations are based) are appreciably less than those between actual ones in the approaching stream, and in the nozzle throat. Again, for convenience in routine use of the curves, magnitudes of the abscissa scale are ones of the <u>ideal</u> (although nominal) magnitudes of the Reynold index.

When subsequent confining of a stream is not required, a nozzle may still be installed with departure from a container or from the end of a pipe, and with discharge to the atmosphere. In such circumstances only the excess of the upstream pressure over the atmospheric need be determined. Or the nozzle may be arranged for passage from the atmosphere into a container or pipe. In both cases special calibration tests may be required, as geometric similarity to the in-pipe arrangement of Fig. 7-2 may not be claimed; but coefficient data on some such arrangements are available in the literature.

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* - Values of the coefficient are ones referred to in NACA Tech. Mem. 952.

(b) Orifice. In the orifice represented in Fig. 7-3, even the converging cone of the venturi tube or the flow nozzle is eliminated, for reasons such as simplicity of manufacture or of installation in an existing pipe line. The device so becomes simply a carefully machined circular aperture in a thin plate, or in a thicker plate but with aperture chamferred on the downstream side. For an in-pipe arrangement access provisions for pressure-difference measurements may be as shown.

In the absence of either the entry cone or any parallel-sided section adjoining it, as at the throat of the venturi tube or flow nozzle, the inertia of those greater portions of the stream which are approaching the orifice other than axially cause persistence of a stream convergence to a minimum area, at its "vena contracta", located downstream from the plane of the orifice. The amount of final convergence (i.e., the minimum stream area) is not readily measurable and also depends on the flow rate.

Pressure-difference measurements made in any manner, as through access points such as indicated, therefore would appear to be of doubly doubtful propriety for use in relations such as equations 7-2, or 2a, with A_2 necessarily interpreted as the fixed and measurable area of the orifice. But again, both dynamic-similarity considerations and experimental evidence indicate that, for geometrically similar installations, discharge coefficients are in fact direct functions of even the nominal (and essentially fictitious) magnitudes of an "ideal" flow rate, as computed by such equations, and of the corresponding Reynolds index. Because of this situation, and of its simplicity, the orifice has become probably the most frequently used flow-metering device in general engineering practice.

The curves of Fig. 7-3 indicate the magnitudes and manner of variation of the discharge coefficient for an in-pipe orifice arrangement such as shown, for a variety of area proportions. The typical appearance of less-than-unity values of the coefficient is evidence simply that the pressure at and transmitted through the down stream connection is more nearly that existing in the stream at its vena contracta, instead of that at the plane of the orifice (of area A_2). It is not evidence of a large energy dissipation in the converging stream, that being quite minor. The ratio of the presumptive area at the vena contracta to the orifice area has been described as the contraction coefficient of the stream, and in this instance is effectively equal to the discharge coefficient.

The orifice may obviously be so installed as to provide for passage of the liquid to or from the atmosphere, in the wall of a container or, respectively, at exit or entry ends of a pipe. The departure of such arrangements from geometric similarity to the in-pipe arrangement makes the discharge coefficients of Fig. 7-3 of questionable applicability, with consequent necessity for calibration tests of a given arrangement or for search of the literature for applicable data.

7-4. Hydraulic Arrester. In the flow through the devices of the several preceeding articles a minimum energy dissipation en route was presumably desired. But in others, such as the recoil cylinder of a gun, the automobile shock-absorber, "dash pots" or vibration dampers, the purpose is instead a controlled energy dissipation. That is their function is to generate, by hydraulic means, a force which will resist relative motion between parts of the device and so serve to transform the kinetic energy of some moving object into internal (molecular) energy in a liquid contained in the arrester.

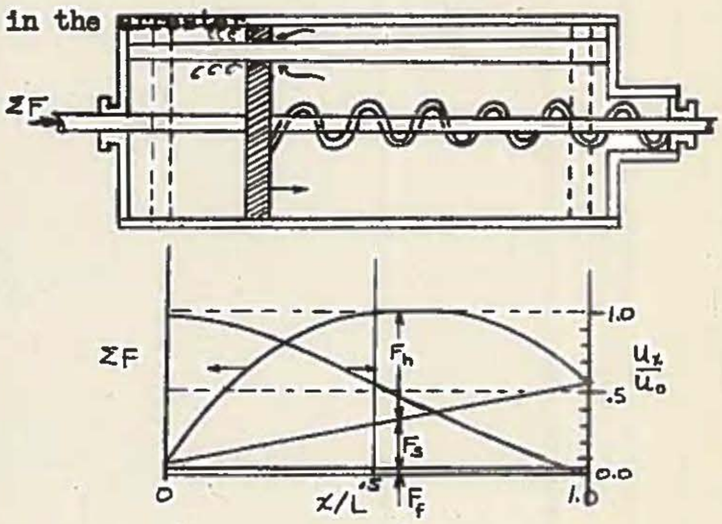

Figure 7-4

The equipment of Fig. 7-4 is reasonably representative of one type*, and its operation is readily analyzed. Essential parts are (a) the piston-rod, through which is transmitted the force produced by the inertia of some moving mass which is to be brought to rest and/or the opposing reactive force, (b) the attached piston, shown with an orifice through it, (c) the surrounding cylinder, which in position and between the heads of which a suitably tapering rod extends through the aperture in the piston, and (d) a suitable liquid, such as oil or glycerin, which fills the cylinder and so must pass through the

* - Others serving like but not at all identical purpose utilize the dynamic forces of Chap 9, in the equivalent of "centrifugal" pumps which are so constructed or operated, however, as to have zero

(annular) orifice from one side of the piston to the other on any relative motion between piston and cylinder. A supplementary item in the figure is the spring, the main function of which is to enable return of the piston to a desired starting position, after any displacememt to the right, although of course contributing to the arresting of the external mass but not dissipating the energy it so absorbs.

Analysis of the actions is aided by reference to the force-distance diagram of the figure, and its supplementary scale and curve (u_x^2/u_o^2) indicating the relative remaining kinetic energy of the external mass as the piston advances in the cylinder. The desired character either of this curve or of the curve representing the total arresting force (ΣF) becomes the index determining in general the proportioning of the arrester components. To discern the relation between these curves observe that

$$\Sigma F = -m\, du/dt \quad \text{and, as} \quad dt = dx/u,$$
$$= -m\, u\, du/dx, \text{ or } -(m/2)\, d(u^2)/dx; \tag{7-3}$$

also
$$\int_0^x \Sigma F dx = -\frac{m}{2}(u_0^2 - u_x^2) = \frac{mu_0^2}{2}\left(1 - \frac{u_x^2}{u_0^2}\right) \tag{7-4}$$

and
$$\int_0^L F dx = \frac{mu_0^2}{2} \tag{7-4a}$$

where ΣF = sum of the forces due to frictional resistance between piston and cylinder (F_f), to compressive stress in spring (F_s), and to the hydraulically generated pressure difference between entry and exit of orifice and corresponding sides of piston (F_h);

m = mass of (external) body the motion of which is to be arrested;

u_o = its arrival velocity;

L = length of piston travel in which its velocity is to have been brought to zero; and

x = any intermediate distance moved by piston.

As graphical analysis is convenient, note that $\int \Sigma F\, dx$ is the area under a ΣF-x curve, that the manner of progressive increase of this area is a direct function of $(u_x/u_o)^2$ and that the total area equals the initial kinetic energy of the mass. The indicated form of the curve would be one causing minimum shock-wise stoppage of the mass and a moderate maximum pressure-loading of arrester parts.

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* - efficiency. Such are the hydraulic dynamometers used for concurrent loading and power-output metering in performance tests of motors or engines.

Relative to the several components of the total decelerative force (ΣF):

(a) The hydraulic components (F_h) and the variables influencing it are related by expressing in two manners the volume-rate of flow of the liquid through the orifice as the piston is caused to advance. That is;

$\dot{V} = u\,A$, where A = net area of piston, excluding orifice,

and also $= C_d a_o \sqrt{\frac{2\Delta p}{\rho}}$ or, as $F_h = \Delta p A$,

$= C_d a_o \sqrt{\frac{2F_h}{A\rho}}$

where a_o = the (variable) area of the annulus formed by the aperture in the piston and the concentric tapered rod through it;

C_d = discharge coefficient for the flow through this annulus, probably variable also;

Δp = pressure difference producing both the velocity of the liquid in passage through the annulus and the force resisting the piston advance;

ρ = density of the liquid.

Thus $$F_h = \frac{A^3\rho}{2}\left(\frac{u}{C_d a_o}\right)^2 \qquad (7\text{-}5)$$

(b) The component due to progressive spring compression, in amount

$$F_s = F_{s,o} + k_s x$$

where $F_{s,o}$ = force due to precompression of spring on installation, with piston in starting position;

k_s = spring constant, dF_s/dx.

(c) The component (F_f) due to frictional resistance between moving parts, probably minor and negligibly variable.

The equilibrium relation associating these forces is, at any position in the piston travel,

$$\frac{A^3\rho}{2}\left(\frac{u}{C_d a_o}\right)^2 + \left(F_{s,o} + k_s x\right) + F_f + \left(\frac{m}{2}\right)\frac{d(u^2)}{dx} = 0 \qquad (7\text{-}6)$$

or in integrated form,

$$\frac{A^3\rho}{2}\int_0^x \frac{u^2\,dx}{(C_d a_o)^2} + \left(F_{s,o}x + \frac{k_s x^2}{2}\right) + F_f x = \frac{mu_o^2}{2}\left(1 - \frac{u_x^2}{u_o^2}\right) \qquad (7\text{-}6a)$$

Integration other than graphical may be quite bothersome. A following example illustrates the use of above considerations in a simpler situation.

Observe that on operation of this device the history of the successive energy transformations includes those from the kinetic energy of the moving mass (1) to work transmitted to the piston facr via the piston rod, (2) to the kinetic energy of the jet generated in the orifice and (3) to wholly haphazard molecular activity in the liquid back of the piston, plus some concurrent dissipation in mechanical friction and some energy storage in the stressed spring. Frequent repetitions of the operation will evidently require provision for accompanying energy extraction from the liquid by some cooling medium.

Example 7-1. A mass of 9000 lbm (280 slugs) moving at 18 ft/sec is to be brought to rest at constant rate of deceleration in a distance of 28 inches. Resistance between moving parts of arrester is relatively negligible, and no spring assists the deceleration. The liquid density is 56 lbm/cu.ft., and for preliminary estimate the discharge coefficient for flow through the annulus may be taken as constant at 0.6.

For these conditions compute the following items.

(a) Energy to be absorbed, requisite force and time interval for stoppage in the specified distance, and the rate of deceleration.
(b) Requisite net area of piston if the pressure difference shall not exceed 800 psi, and the piston diameter if the diameter of the aperture through piston is 13/16" and that of the piston rod is 1.20 inch.
(c) Proper diameters of the tapering rod forming the annulus, at 0.0", 7", 14", 21" and 28" from starting position of piston.
(d) Temperature rise of liquid, following one operation, if its total volume is 50% in excess of that displaced, its specific heat is 0.5 btu/lbm, °F, and no heat is conducted to environs.

Solution.-

(a) Energy absorbed $= 280 \times 18^2/2 = 45{,}360$ ft. lbf;
Force $= 45{,}360/2.333 = 19{,}440$ lbf, and constant;
Deceleration rate $(= F/m) = 19{,}440/280 = 69.43$ ft/sec^2;
Time $(= \sqrt{2L/\text{accel.}}) = \sqrt{2 \times 2.333/69.43} = 0.26$ sec.

(b) Net piston area $= 19{,}400/800 = 24.30$ sq. in. or .1687 sq. ft.;
Total area $= 24.30 + (\pi/4)(.8125^2 + 1.20^2)$
$= 24.30 + 0.52 + 1.13 = 24.95$ sq. in. and
Diameter $= 5.75$ inch.

(c)

x, (inches / feet)	0.0" / 0.0 ft	7.0" / .5833	14.0" / 1.167	21.0" / 1.750	28.0" / 2.333
Residual kinetic energy of mass	45,360 ft. lbf	34,020	22,680	11,340	0.0
Residual velocity (u)	18 ft/sec.	15.60	12.73	9.00	0.0
$a_o, = \frac{144\,u}{C_d}\sqrt{\frac{A^3 \rho}{2 F_h}}$	0.201 sq in	0.174	0.142	0.100	0.0
$a_{rod}, = 0.518 - a_o$	0.317 sq in	0.345	0.376	0.418	0.518
Diameter of rod	0.636 in	0.663	0.692	0.729	0.812

(d) $$T, = \frac{45{,}360/778}{0.5(.1687 \times 2.333 \times 1.5 \times 56)} = \frac{58.3 \text{ btu}}{16.8 \text{ btu}/°F} = 4.1°F.$$

7-5. Flow of Gases through Venturi Tube, Flow Meter or Orifice.

The energy relations applying to the flow of expansible fluids through these devices are in principle the same as those for liquids, and relations such as those of equations 7-2 or 7-2a may in fact be employed if the pressure change en route through them does not exceed about 1 per cent of the absolute upstream pressure. But, with the greater pressure changes accompanying higher flow rates, the requisite accounting for density decrease with progressive pressure decrease necessitates the use of more complex relations, and also the awareness of limitations to the flow-rate capability of these devices. The reader is referred to thermodynamic literature for more adequate attention to these considerations.

For present purposes it is sufficient to quote the following relation, which parallels equation 7-2a in the sense of associating the flow density (G) with the pressure change in a converging stream but accounts for the progressive density decrease en route; that is

$$G_{1,\text{ideal}} = 2.646 \frac{A_2}{A_1} \frac{p_1}{\sqrt{RT_1}} \sqrt{\frac{\left(\frac{p_2}{p_1}\right)^{1.429} - \left(\frac{p_2}{p_1}\right)^{1.714}}{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{1.429}}} \tag{7-7}$$

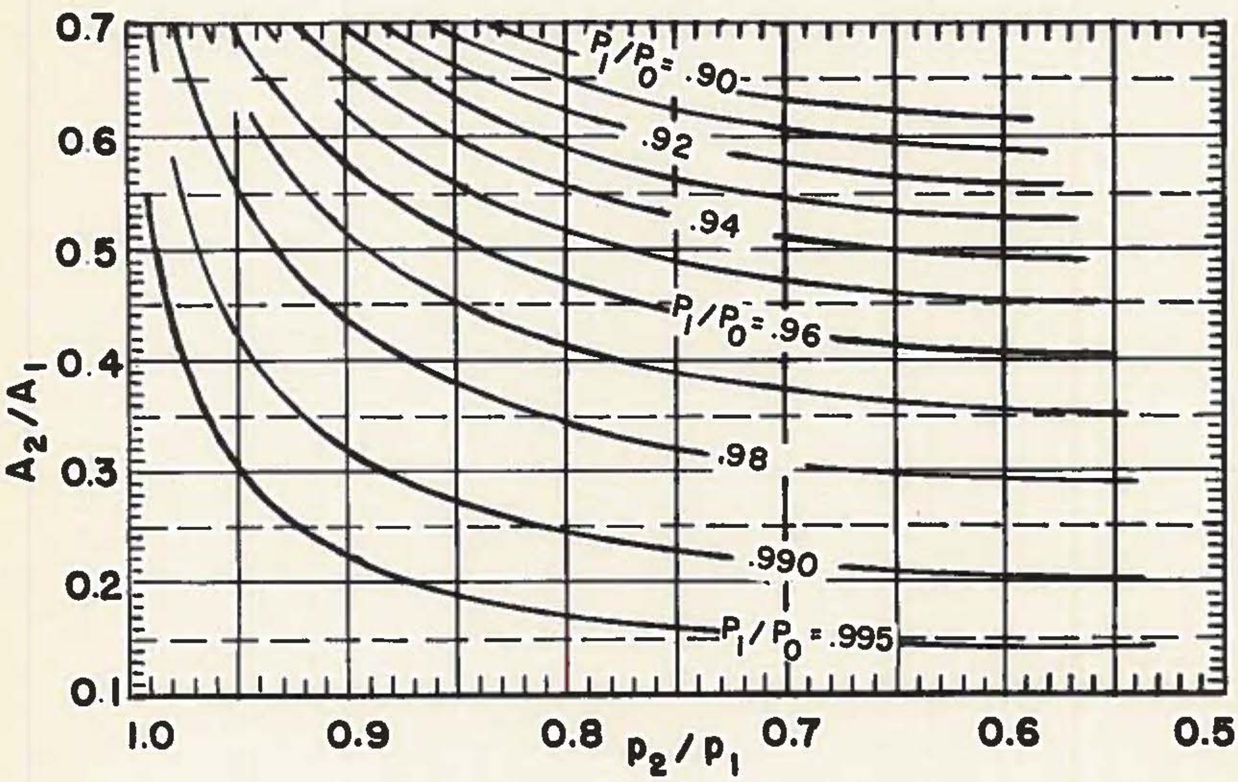
for G in slugs/(sec, sq ft), and R in ft lb/(slug, °R) or 1716 for air. The numerical constant and exponents are suitable, however, only for diatomic gases at moderate pressure and temperature levels. These are necessarily expressed in terms of their <u>absolute</u> magnitudes. Adequately exact evaluations of the term $\left[(p_2/p_1)^{1.429} - (p_2/p_1)^{1.714} \right]$ of this relation may be bothersome when p_2/p_1 approaches 1.0. But at values of $(p_1 - p_2)/p_1$ not exceeding about 0.1 it may be evaluated with minor approximation by the alternative expression

$$0.286(p_1 - p_2)/p_1 - 0.306 \left[(p_1 - p_2)/p_1 \right]^2 \tag{7-7a}$$

Limitations to the applicability even of equation 7-7 result from an inherent inability of gases to expand, in converging streams or channels such as those of the devices under consideration, to pressures less than about 1/2 of the absolute upstream pressure. The maximum attainable through-flow for these is thereby limited, or they become "choked" at that flow-rate, even if inflow were from a larger container and the relative pressure in the region to which they are discharging were in any manner caused to be less than the above 1/2. For the diatomic gases this maximum rate is such that, for flow from a large container,

$$G_{2,\ \text{max.ideal}} = 0.685 \frac{p_0}{\sqrt{RT_0}} \tag{7-8}$$

It will be recalled that a like situation was reported (Art. 6-12) as existing in the flow of expansible gases through constant-area channels. Both relate to ultimate attaining of sonic velocity (i.e., unity Mach number.)



p_1/p_o Correlation, Diatomic Gas Streams

Fig. 7-5

A graphical facility providing an alternative to the use of equation 7-7 is furnished by Figure 6-6 when supplemented by Figure 7-5. The procedure for the use of these is as follows.

(1) Enter Figure 7-5 at the observed magnitudes of p_2/p_1 and A_2/A_1, and read the accompanying value of p_1/p_o;

(2) Compute thereby the value of p_2/p_o and interpreting the abscissa scale of Figure 6-6 as one of p_2/p_o, enter that figure and pass to the line for $fL/D = 0$;

(3) Read the corresponding values of G/G^*, which is now to be interpreted as G_2/G^* ($= G_1A_1/G^*A_2$), and compute G_2 and/or G_1.

It will be observed that the above "choking" situation is reflected in Figure 6-6 by the termination of the line for $fL/D = 0$ at a value of $p_2/p_o = 0.528$.

Equation 7-7, or the graphical facilities, serve in effect to account for the influence of variation of the Mach number in the flow of expansible fluids in convergent streams. Influences of the Reynolds index, and of the geometric character of the flow channel and location of the access points for pressure measurements in in-pipe installations, were indicated in the foregoing for several flow-metering arrangements. For the flow of gases through these devices the flow-rates are commonly sufficiently high that change in the magnitude of the Reynolds index has minor effect.

The following example illustrates the use both of equation 7-7 and of the above graphical facilities.

Example 7-2. In the passage of dry air through an orifice of 4.25" diameter in a 6" pipe, of form and in-pipe installation such as that of Figure 7-3, an upstream pressure of 90 psia and temperature of 80°F are observed, and also a pressure difference of 15.0 inches of mercury as between upstream and downstream pressure connections.

For these conditions determine the following items.

(a) The (nominally) ideal rate of flow through the pipe, in lbm/minute, by the use of equation 7-7. But for general interest compare with this result ones obtained by use of the alternative expression quoted in connection with that equation, and on introducing the upstream density in equation 7-2a.

(b) The ideal flow rate as found by the use of Figure 6-6 and 7-5.

(c) The Reynolds number (D_1G_1/μ) corresponding to the ideal flow rate, and, selecting a suitable value of C_d, the actual flow rate.

Solution. General items. $D_1 = 0.5$ ft; $A_1 = (\pi/4)(6/12)^2 = .1964$ sq. ft; $A_2/A_1 = (4.25/6.0)^2 = .502$; viscosity of air at 80°F = (by Fig. 1-2) 3.6×10^{-6} slugs/(sec, ft); ρ_1, at 90 psia and 80°F, $= (90\text{x}144)/(1716\text{x}540) = 0.014$ slugs/cu.ft; $p_2/p_1 = 1 - (15.0\text{x}.491)/90 = 1 - .0818 = .9182$, and also $\frac{p_1}{\sqrt{RT_1}} = (90\text{x}144)/\sqrt{1716\text{x}540} = 13.47$ (sec, sq. ft.)/slug.

(a) At $p_2/p_1 = .9182$, $(p_2/p_1)^{1.429} = .8851$, $(p_2/p_1)^{1.714} = .8638$, and their difference = .0213. Also $1 - (A_2/A_1)^2(p_2/p_1)^{1.429} = .779$. Thus G_1, ideal, $= 2.646 \times .502 \times 13.47 \times \sqrt{.0213/.779} = 2.96$ slugs/(sec, sq.ft.), and $\dot{m} = (2.96 \times .1964)(32.17 \times 60)$ $= 1.120$ lbm/min.

For comparison:- $0.286 \times .8182 - .306 \times (.8182)^2 = .02135$, and agrees well with the above .0213, as $(p_1 - p_2)/p_1$ is less than .10; and, by eq. 7-2a, G_1, ideal, $= .014 \sqrt{2 \times (15.0 \times .491 \times 144/.014)/(1/.502^2) - 1}$ $= 3.14$, and does not agree well with the above 2.96, as $(p_1 - p_2)/p_1$ considerably exceeds .01.

(b) By Fig. 7-5, at $p_2/p_1 = .918$ and $A_2/A_1 = .502$, $p_1/p_o = .976$, $p_2/p_o = .918 \times .976 = .896$; and $p_o = 90.0/.976 = 92.2$ psia or 13,280 psfa.

In Figure 7-5, at $p_2/p_o = .896$ and $fL/D = 0$, $G_2/G^* = .630$; also, with $p_1/p_o = .976$, $T_1/T_o = .995$ and $T_o = 543°R$. With $G^* = (.685 \times 13{,}280)/\sqrt{1716 \times 543}) = 9.42$, $G_1 = (.630 \times 9.42) \times .502 = 2.98$ slugs/sec, sq. ft.) and agrees reasonably with the above .296

(c) By Fig. 7-3, at $D_1G_1/\mu = .50 \times 2.96/(3.8 \times 10^{-6}) = 3.9 \times 10^5$ and $A_2/A_1 = .502$, $C_d = .696$ and thus $\dot{m}$, actual, $= 1{,}120 \times .696 = 780$ lbm per minute.

Other types of devices are available and useful for rate-of-flow metering in closed channels. One is analyzed in the following article. Another, employing simply a pipe bend or elbow, and facilities for measuring a relevant pressure difference, is considered (art. 9-8) after the principles on which its utility depends have been given requisite attention.

7-6. Variable-area Flow Meters. Recall that in devices such as the venturi, nozzle or orifice, with built-in and constant areas through which the flow is directed, rate-of-flow evaluation is made by supplemental measurement of a pressure difference which is to a degree proportional to the square of that rate. Due to this, and to the marked variation of the discharge coefficient at lower magnitudes of the Reynolds index, the obtaining of adequate precision in measurement of the pressure, and thereby of the flow rate, may be troublesome at lower rates.

In contrast to this, a type of device has been developed more recently in which flow is effected by a pressure difference which is ingeniously maintained constant, but measurement of which is not required. Indirect but easy determination is, however, made of the aperture area required for constants for a given float and fluid. It is of interest in this connection that, if $\rho_f = \rho_w$, moderate variation of the density of the fluid may be shown to influence rather trivially the item $(\rho_f - \rho_w)\,\rho_w$, and thus, further, the mass-rate of flow $(\dot{m})$ signified by a given position of the float. Oppositely, the volume-rate $(\dot{V})$ significance is less influenced (but still quite appreciably) by variation of the fluid density when the ratio ρ_f/ρ_w is much greater. Such greater magnitudes of the ratio are typical if the fluid being metered is a gas.

The general nature of the flow pattern en route past the float is indicated in the figure. Although there is some energy dissipation due to turbulence above the float, this has a beneficial feature in causing very moderate change in the discharge coefficient (C_d). Although quite adaptable to flow measurements at high flow rates, this type of device is particularly well adapted to the precise determination of very low rates.

7-7. Velocity Meters; Impact and Allied Tubes. Instruments of reasonable precision are available for the direct measurement of the velocity of one dimensional streams of some breadth. These include the anemometer or the mariners' log which take the general form of a shaft with vanes or foil-shaped radial blades about which the stream passes and causes their "windmilling" and thereby the actuation of a revolution counting mechanism. For the measurement of velocity inside tubes, propeller driven, electromagnetic or radioactive tracer devices have been used. In addition, wide application has been found for the hot wire anemometer, an electrically heated wire whose temperature and therefore resistance is extremely sensitive to the velocity of the fluid passing over it. The electrical method is particularly well adapted to the measurement of rapidly fluctuating velocity, as in highly turbulent streams.*

In those many instances where it is desired to determine the magnitude

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* - See Goldstein, "Modern Developments in Fluid Dynamics", Vol. I or Prandtl and Tietjens, "Applied Hydro and Aero-Mechanics" for further discussion of some of these instruments.

of the velocity at a selected point in a stream (i.e., in a particular stream tube, Art. 5-2.), or its distribution in an entire stream, a simple device known as an *impact* or *Pitot* tube or a modification thereof has found the most wide-spread use. The major element is merely a small diameter tube with the open end facing directly into the stream. No flow through the tube occurs, however, as the other end connects to a suitable pressure-measuring device. The particles in that stream tube which is concentric with the opening of the impact tube are caused to decelerate momentarily to zero relative velocity as they approach the opening, but then are diverted about the tube end and ultimately re-acquire their prior velocity. The pressure developed at the tube end and transmitted to the pressure sensing device, is known variously as the *impact*, *stagnation* or *total* pressure.

In actual practice it is usual to determine the excess of the stagnation pressure over that of the undisturbed stream, known as the *free-stream* or *local* pressure, and also, although perhaps inaptly, as the "static" pressure. The difference between the stagnation and local pressure is referred to as the *kinetic*, *dynamic* or *velocity* pressure.

The static pressure may be measured at a small aperture in the wall of the confining channel (carefully machined so as not leave any protuberance on the inside surface of the channel) as in Figure 7-7; or more conveniently, by an arrangement known as the *Pitot-static* tube. Here the impact tube has about it a concentric tube which is open to the fluid only at several small apertures placed in the parallel portion of its wall, and perfectly normal to the wall. In a tube designed by Prandtl a slit in place of the apertures is provided downstream of the tube end, yet upstream of the handle sufficient distances so

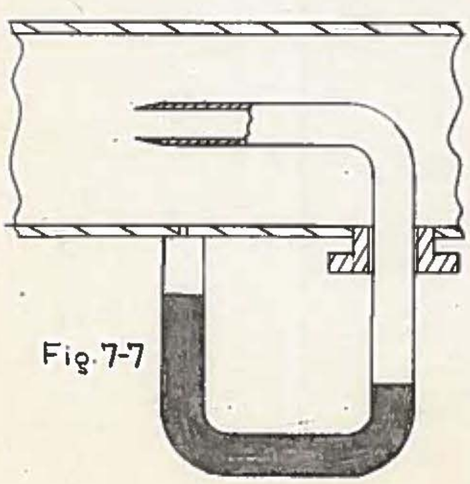

Fig. 7-7

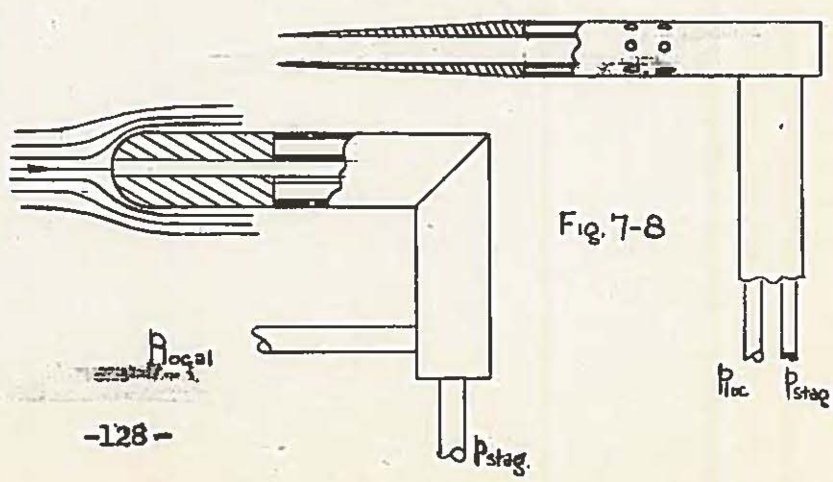

Fig. 7-8

that the effects of the two disturbances to the fluid flow balance each other, and the true local pressure is transmitted through the annular portion of the double tube. Figure 7-8 illustrates two Pitot-static tubes having slightly different characteristics, and a typical pattern of the stream lines in the stream as it approaches, reaches and passes the impact tube. The energy equation for that stream tube which stagnates at the impact tube opening, when expressed in terms of its initial or free-stream velocity, u , becomes simply

$$u = \sqrt{\frac{2(p_{stag.} - p_{local})}{\rho} + \phi} \tag{7-11}$$

or

$$u = C\sqrt{\frac{2(p_{stag.} - p_{local})}{\rho}} \tag{7-11a}$$

where C is a coefficient which accounts not only for the energy dissipation, ϕ , but for deviations between the true and measured values of the two pressures. For well designed instruments the value of C differs from unity by less than 1%. However, in highly turbulent flow, velocity components in the transverse direction cause the static pressure indication to be high by several per cent, due to the impact influence on the static pressure apertures.

The Pitot-static tube is not well adapted for the determination of the direction of flow, or alternatively for verifying that its axis is oriented in the direction of approach of the stream. Indeed, the relative independence of the pressure indications with respect to the angle of orientation, for 5° to 10° from the direction of the velocity, is a characteristic intentionally designed into the instrument. For ascertaining the direction as well as the magnitude of the velocity, the impact tube end may be located in a small sphere, in the wall of which there are, however, a pair of symmetrically located apertures through which local pressures are transmitted to a pair of pressure-difference indicators. The stream lines of Figure 7-8 suggest (and analyses of Chapter 13 verify) that the flow pattern about the sphere is such that the velocity and pressure at some particular angular position are the same as those in the undisturbed stream. Thus, by locating the pair of apertures in the sphere surface at the proper positions and so orienting the instrument when in use that the difference between the stagnation pressure and each local pressure is identical, both the direction and magnitude of velocity are determinable.

Various other devices for the measurement of flow direction have been used, and many with greater precision than that described above.*

If the volume-rate of flow is to be determined by impact tube in a stream of some transverse area but non-uniform velocity distribution, it becomes necessary that local velocity measurements be made at a number of points at which the velocity is representative for a stream segment of known transverse area. The aggregate volume rate is evidently the sum of the products of each area times the velocity component normal to it.

For flow through a circular duct, an equivalent procedure is to plot the local velocities, as ascertained at a number of points across the stream, against the square of the radial distance to those points. The mean velocity of the stream is the mean ordinate of such a graph. Figure 7-9 illustrates. The validity of accepting this mean as representative for the entire stream is seen by noting that

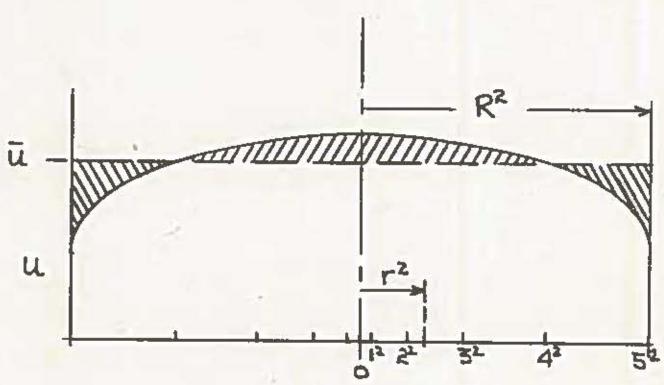

Figure 7-9

$$\bar{u}, = \frac{\dot{V}}{A}, = \frac{1}{\pi R^2}\int_{-R}^{R} \pi u r\,dr = \frac{1}{2R^2}\int_{-R}^{R} u\,d(r^2) \qquad (7\text{-}12)$$

where r = radial distance to any point, and
R = radius of duct.

If the fluid is a compressible gas and velocities are subsonic but the pressure rise to stagnation is sufficient to affect the density materially, thermodynamic analysis establishes that, for diatomic gases, velocities are

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* - See footnote on page 127, as well as publications of the National Advisory Committee for Aeronautics (NACA), e.g. TN 2830.

expressed by the relation

$$u = 2.643\sqrt{R\,T_o\left[1 - (p_{local}/p_{stag})^{.286}\right]} \qquad (7\text{-}13)$$

where T_o is also an (absolute) "stagnation" temperature, as measured perhaps by thermocouple located at a stagnation point produced in the stream, and again R for air is 1716 ft.lbf/(slug, °Fabs).

7-8. Problems -

1. (a) An 8" x 4" Venturi meter with vertical axis, through which fresh water at 50°F is passing shows a differential manometer reading of 20 inches of mercury. Water fills the connection to the manometer. Estimate the rate of flow in cubic feet per second and the throat velocity.

(b) What is the correction factor for initial velocity for the meter, and what will be the flow coefficient (C) corresponding to the estimated flow rate?

2. (a) Using the data and results of problem 7-1 compute the value of Ø for the meter nozzle and also the parameter $\frac{2\phi}{u^2}$ (i.e., the nozzle resistance factor, art. 6-8).

3. For the meter and manometer reading of problem 7-1 estimate the flow rate if the fluid were instead oil of 600 centipoises viscosity and 0.950 specific gravity?

4. For a flow nozzle of the same proportions as the Venturi tube of problem 7-1, estimate the flow rate if pressure readings at the throat and one pipe-diameter upstream from the nozzle gave the same differential manometer reading of 20 inches of mercury.

5. For a sharp-edged orifice of 4 inch diameter located centrally in an 8 inch line, estimate the flow rate if a differential manometer connected to corner taps read 20 inches of mercury and the fluid were (a) fresh water at 50°F, (b) the oil of problem 7-3.

6. For a 2-1/2" by 7/8" fire nozzle, delivering fresh water and delivering a jet of the latter diameter, estimate the jet velocity, the rate of delivery, efficiency and power if the pressure at the hose connection was 50 psi gage at a point 1 foot below the exit end of the nozzle.

(b) Considering the horizontal and verticle components of the jet velocity, compute the height to which the stream might ideally be delivered if directed at an angle of 75° with the horizontal, the corresponding time interval for a parcel of water leaving the nozzle to arrive at the top of the trajectory, and the corresponding horizontal distance (ideal) from the nozzle to trajectory summit.

7. At radial locations as indicated below in a 10" pipe readings as shown were observed on a differential mercury manometer connected between impact and static connections of a Pitot-static tube. The fluid flowing through the pipe and filling the manometer connections is water. As the traverse was found to be effectively symmetrical, data for only one half of a diameter are given.

Distance from center, inches	0.0	1.0	2.0	3.0	3.5	4.0	4.5	4.75	5.0
Reading, inches of mercury	1.78	1.73	1.61	1.43	1.33	1.17	0.97	0.81	(zero)

(a) By computing the individual velocities and plotting suitably, determine the mean velocity ($\bar{u}$) and the volume-rate of flow $\dot{V}$ in the pipe.

(b) Determine the radial location at which the mean velocity might be determined by a single reading, and also the ratio of the mean to center-line velocities.

8. Observe that the rate of kinetic energy passage through a circular pipe may be expressed as $\sum\left(\frac{\dot{m}u^2}{2}\right)$ and thus as the integral

$$\int_0^R \rho u(2\pi r\,dr)\frac{u^2}{2} \quad \text{or} \quad \pi\rho\int_0^R u^3\,d(r^2).$$

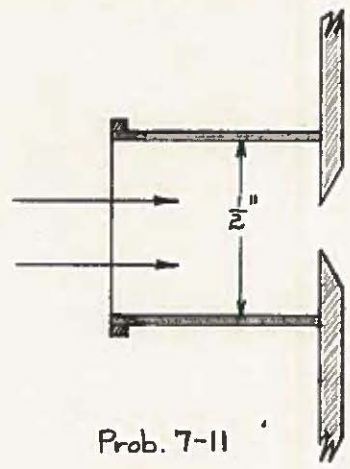
Using the velocity and flow-rate values found in problem 7-7 draw a graph of u^3 vs r^2 and, from its area or mean height, determine the rate of kinetic energy passage past any section of the pipe (ft.lb/sec), and thereby determine the ratio between the true kinetic energy per unit mass and the nominal value as expressed conventionally by the item $\bar{u}^2/_2$. Compare the ratio as so found for the present (turbulent) flow conditions with that for laminar flow (art. 6-3.)

9. A variable area rotameter is calibrated for water of density 1.94 slugs/ft^3 and has a float of density twice that of water. Calculate the % of error in $\dot{m}$ and $\dot{V}$ when oil of specific gravity 0.88 is being metered with this instrument.

10. A large pipe line carrying water at 50 psig develops a hole approximately 0.2" diameter. Assuming that the hole behaves as a sharp edged orifice, estimate the rate of water leakage, and the height to which the jet will rise

if directed vertically upward.

11. In order to insure a steady flow of air from the atmosphere into a tank whose pressure fluctuates between 4 and 7 psia, a small orifice is to be placed in the side of the tank as shown, surrounded by a short 1/2" screened nipple to prevent foreign matter from clogging the orifice. For atmospheric temperature of 80°F, what diameter orifice will allow an air flow of 10^{ft3}/min of free air.



Prob. 7-11

PART IV

FLOW DYNAMICS

Chapter 9. General Considerations, Flow Dynamics

9-1. Foreword. The considerations and the engineering tools developed in Part 3, namely the adaptation of the principles of the conservation of energy and of mass and the similarity concepts to fluid flow conditions, enable informative analysis of test performances of existing devices, and the utilization of such information for anticipating the performance of geometrically similar devices. These are most useful accomplishments. But it is even more essential and more difficult to so design a new machine that it be capable of accomplishing efficiently desired energy transformations, or to predict the behavior of an untested one.

For such purposes one must be aware of the manner in which forces may be applied to or generated by fluid streams. Studies in which emphasis is on force aspects of flow are known as those of fluid dynamics, and are the major concern of this Part IV. These analyses will be facilitated by some preliminary attention to kinematic aspects of flow (i.e., space-time relations), but a more complete development of these is deferred to Part V.

9-2. A Fluid Dynamics Law. In the somewhat approximate methods of Chapter 5, transverse distributions of energy in a fluid stream were recognized but averaged in order to apply the Conservation of Energy Principle in the direction of flow. However, with respect to the variation of properties and energies normal to the direction of flow, one quite fundamental tendency requires recognition. Its basic nature justifies its classification as a nuatural "law" of fluid dynamics. It may be expressed as follows.

In any fluid stream an influence invariably operates which tends to establish a transverse equi-distribution of the aggregate mechanical energies; that is, as regards the particles forming the stream lines and tubes which make up the flow field, the sum of their geopotential, kinetic and flow-work energies tend to be the same at all positions in a plane normal to the direction of flow.

An associated consideration, and one which is so intimately related that it may be regarded as a corollary of this law, is as follows.

With real fluids this tendency is invariably hampered, in greater or less degree, by the influence of the viscosity of the fluid (possibly with consequent macroscopic orders of turbulence.) The influence is of major order when flow conditions are such that viscous forces predominate over inertial, but less when inertial forces are dominant.*

An excellent example of the law and its corollary is afforded in the phenomenon of the transition of a stream from uniform to laminar and thence to turbulent flow by gradual increase of the Reynold's index. Referring to Figure 9-1, for a constant mass rate of flow the Reynold's index, N_R,

$$= \frac{DG}{\mu} = \frac{D\dot{m}}{\pi \frac{D^2}{4} \mu}$$

is inversely proportional to the diameter, so that in the situation pictured N_R gradually increases downstream. Near the entrance region of the pipe (a) the flow and energy distribution are nearly uniform, conforming to the above fluid law. As laminar flow is established (b), viscous forces predominate and the uniformity of kinetic energy is destroyed (this corresponds to the corollary). However, as N_R increases, the energy distribution soon becomes sufficiently non-uniform to cause a re-distribution, manifested by the shift to turbulent type flow (c). This change is direct evidence of the re-assertion of the fluid law in the region where

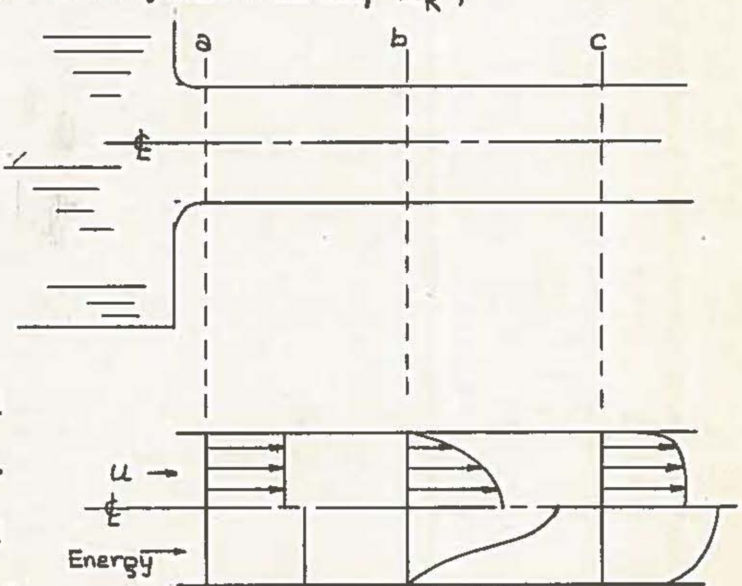


Figure 9-1

* - These considerations are quite analogous to or, in a sense, mechanical adaptations of the Second Law of Thermodynamics. By those familiar with the various aspects of that Law it will be recalled as stating, from a more philosophical viewpoint, (a) that there is an invariable trend in all processes in nature to an arrival at a common energy level, but (b) that the intensity of this trend is influenced by effects known as irreversibilities, there including ones both of mechanical and of thermal character. Viscous influences, frequently described alternatively as ones due to fluid "friction", are in the first category. Discussions such as ones in "The Mysterious Universe", Eddington, or in "Principles of Engineering Thermodynamics", Kiefer, Kinney and Stuart, emphasize these considerations.)

inertial forces are again dominant.

Following articles consider like situations in the curvi-linear flow of the free, forced and viscous vortices.

9-3. Sources of Dynamic Forces. In the earlier periods of development of power or pumping equipment requisite forces were produced merely by a direct action between a compressed fluid in a cylinder and the face of a moving piston. For the most part present-day equipment depends on the dynamic forces generated by change in both the magnitude and the direction of the velocity of a stream which is caused to pass through suitably devised channels located on a rotor, this being connected to an exterior rotating shaft. The energy requirements for driving the ship, and its propellor, originate similarly in the forces associated with changes produced in the state of motion of the environmental fluid.

The dynamic forces associated with velocity change in a stream are attributable to the accompanying pressure changes. They may best be correlated by application of Newton's laws of motion to a discrete and differentially small particle of any fluid forming the stream. Referring to Figure 9-2, for recognizing primary aspects of the relevant forces on the particle it is sufficient for present purposes to limit attention to a two-dimensional stream that is there represented as proceding in a curved path in a vertical plane (i.e., to r and s coordinates), and to a particle in the stream which is momentarily located at a specific point and is of dimensions dr, ds and depth dy normal to the plane.

Motions in the stream will be presumed to be steady (i.e., invariant with time), so that successive particles arriving at the point exhibit like motion. The motion of the particle is momentarily that about an instantaneous center O, at radial distance r. The dimension ds (in the direction of motion) may be regarded also as the distance the particle would advance in that direction in time dt.

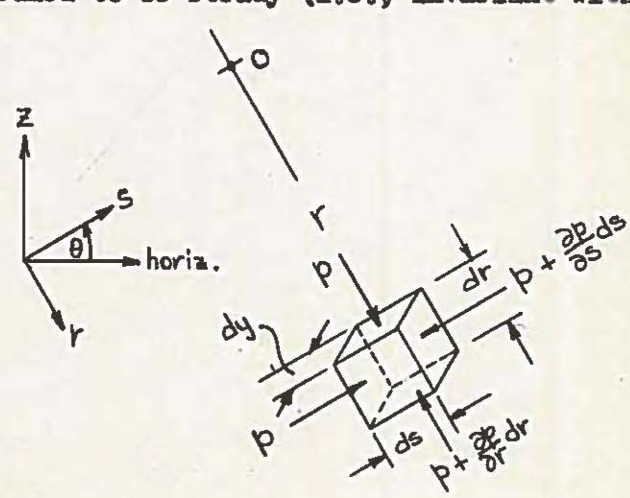


Figure 9-2

Its linear velocity, denoted as u_t, is thus ds/dt, but is also expressible as ωr, where ω is the angular velocity of the particle about 0. With density ρ the particle has mass dm equal to ρ (dr ds dy). Motions and forces in the direction of linear advance and radially outward will be regarded as positive in sense. Angle θ is that between the direction of motion and the horizontal.

Considering initially the forces associated with possible change in linear velocity, by acceleration in the tangential direction, such will include external or body forces due to a pressure difference as between upstream and downstream faces of the parcel, to the tangential component of the gravitational force, and to frictional restraint between adjacent streams of particles or between stream and boundary. Any resultant acceleration (or deceleration) is opposed by the inertia of the particle. These may be expressed specifically as follows.

(a) Pressure force on particle in amount

$$dF_{p,t} = p\,dr\,dy - \left[p + \frac{\partial p}{\partial s}ds\right]dr\,dy = -\left(\frac{\partial p}{\partial s}\right)ds\,dr\,dy$$

(b) Gravitation force, in amount

$$dF_{g,t} = -g\,dm\sin\theta \text{ and, from Fig 9-2, } = -g\rho(dr\,dy\,ds)\frac{\partial z}{\partial s}$$

(c) Frictional restraint, and expressed adequately for present purposes simply by symbol $dF_{f,t}$.

(d) Inertial restraint, $dF_{i,t}$, in amount dm a_t, or - dm du_t/dt as velocity increase in the direction of motion is opposed by inertial restraint. Or, as u_t = ds/dt, and expressing dm in terms of attributes of the parcel,

$$dF_{i,t} = -dm\frac{du_t}{dt} = -\rho(dr\,ds\,dy)\frac{du_t}{dt}$$

$$= -\rho\,dr\,dy\,u_t\,du_t\text{, or } -\rho\,dy\,dr\,d(u_t^2)/2$$

The sum of the body forces as expressed with due attention to their directional significance may be equated to the mass-times-acceleration, or the sum of the body forces and the inertial restraint may be set equal to zero. Following the latter procedure

$$-\left(\frac{\partial p}{\partial s}\right)ds\,dr\,dy - g\rho(ds\,dr\,dy) - dF_{f,t} - dr\,dy\frac{d(u_t^2)}{2} = 0$$

or dividing by $-\rho\,dr\,dy$ $(= dm/ds)$ and integrating

$$\int \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)ds + g\int\left(\frac{\partial z}{\partial s}\right)ds + \int \frac{dF_{f,t}\,ds}{dm} + \int \frac{d(u_t)^2}{2}ds = \text{constant} \quad (9\text{-}1)$$

Here the F_f-term will be recognized as expressing the energy dissipation ($d\phi$) per unit mass in movement through distance ds. If a density variation is to be accounted for, this relation may be written as

$$\int \frac{1}{\rho}\frac{\partial p}{\partial s}ds + gz + \phi + \frac{u_t^2}{2} = \text{Constant} \quad (9\text{-}1a)$$

or, for an incompressible fluid, as

$$\frac{p}{\rho} + gz + \phi + \frac{u_t^2}{2} = \text{constant} \quad (9\text{-}1b)$$

Although the similarity between this relation and the steady-flow energy equation is meaningful, that provides for the accounting of outside influences while these do not. But, through their accounting of the individual forces acting in the direction of travel of the particles forming a stream tube, they serve to relate specifically the pressure, gravitational and the net accelerative forces acting on any particle as it travels along a stream line.

A second equation is necessary for associating such variables when the direction normal to the flow are of concern. To develop it consider the forces associated with change in direction of travel or <u>radial acceleration</u> of the particle, is of more direct present concern. The individual forces participating may be expressed as follows. Frictional influences are disregarded as no radial relative motions are involved.

(e) Pressure force on particle, associated with any pressure difference between radially inner and outer faces, and in amount

$$dF_{p,r} = p\,ds\,dy - \left[p + \left(\frac{\partial p}{\partial r}\right)dr\right]ds\,dy = -\left(\frac{\partial p}{\partial r}\right)dr\,ds\,dy$$

Gravitational force, in amount

$$dF_{g,r} = g\,dm\cos(\pi-\theta) = -g\rho(dr\,ds\,dy)\left(\frac{\partial z}{\partial r}\right)$$

(g) Inertial (or "centrifugal") force, due to radial acceleration, in amount basically equal to <u>dm ar</u> but commonly written as*

$$dF_{i,r} = dm\, u_t^2/r, \quad = \rho(dr\,ds\,dy)\frac{u_t^2}{r} \quad \text{or } \rho(dr\,ds\,dy)\,\omega^2 r$$

Again equating the sum of these forces to zero, maintaining due attention to directional significances,

$$-\left(\frac{\partial p}{\partial r}\right)(dr\,ds\,dy) - g\rho(dr\,dy\,ds)\left(\frac{\partial z}{\partial r}\right) + \rho(dr\,dy\,ds)\frac{u_t^2}{r} = 0$$

or, dividing through by $-\rho(dr\,dy\,ds)$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\right) + g\left(\frac{\partial z}{\partial r}\right) = \frac{u_t^2}{r}, \text{ or } \omega^2 r \tag{9-4}$$

The immediate significance of this relation lies in its indication of the invariable trend to pressure increase with increasing radial distance across a stream tube composed of an assembly of curving stream lines. In fact many of the devices with which the engineer is currently concerned accomplish their objectives in part through strategic utilization of this consideration.

Equation 9-4 is capable of integration, across such a stream tube, only when the manner of variation of the tangential velocity, with respect to the radial direction, is known or may be specified.

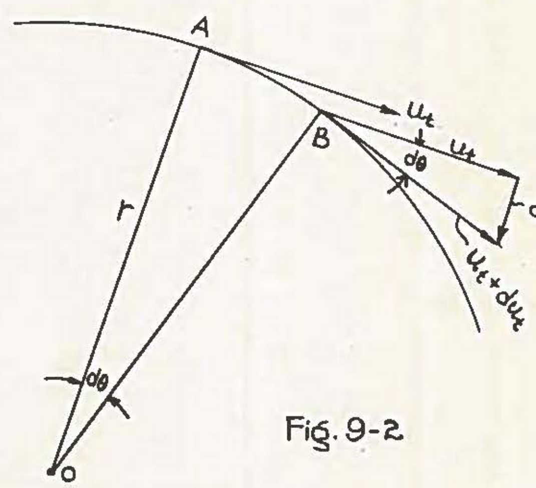


Fig. 9-2

* – Figure 9-2 represents successive but infinitesimally separated positions of a particle in a stream line in passage about instantaneous center O, from a position at A at which it has velocity u_t to a position at B at which it has velocity $u_t + du_t$. Indicating significant components of velocity $u_t + du_t$ by the vector triangle at B, it is seen that the radial component, du_r, of the velocity change du completes an triangle with two legs representing u_t, and $(u_t + du_t)$.

This triangle is further similar to triangle OAB. From their similarity,

$$\frac{du_r}{ds} = \frac{u_t}{r}, \text{ or } du_r = \left(\frac{u_t}{r}\right)ds \tag{9-2}$$

$$\text{and thus} \qquad a_r = \frac{du_r}{dt} = \frac{u_t}{r}\frac{ds}{dt} = \frac{u_t^2}{r} \tag{9-3}$$

In many instances such integration may be exceedingly complex. However, it will be informative to do so in the following for several simpler but conventionalized types of flow pattern.

9-4. Vortex Flow-Configurations. The general term vortex configuration, as it relates to a moving mass of fluid, denotes a character of motion in which a major component of such motion is a travel of the individual particles and an orientation of the stream tubes in circular paths about a center O. In engineering equipment other motion components are normally superimposed, such as an accompanying radial travel, but the flow patterns may still be said to be of vortex character. These are classified into the primary categories of (a) free, (b) forced and (c) viscous vortices, and the supplementary ones of (d) spiral and helical vortices in which radial motion is superimposed on the free or forced vortex.

9-5. Free, or Potential, Vortex. This configuration is described as free as it is the one to which any vortex-type of flow pattern tends spontaneously to revert, seemingly as a result of the efforts of Nature to produce an equilibrial situation in which the aggregate mechanical energy of all particles in the flow field is the same.* The name potential arises from certain kinematic conditions discussed in chapter 13. Except as viscosity influences prevent its full attainment with real fluids, it is characteristic of cyclonic disturbances in the atmosphere, liquid "whirl-pools" et cetera.

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* - This action is readily demonstrable. For example, if attempt has been made to enforce another type of configuration, as by active rotation of a "paddle wheel" immersed and located axially in an enclosing glass cylinder which is partially filled with water, on quick withdrawal of the rotor the water promptly and spontaneously reverts to effectively the free-vortex configuration, except as dominant viscosity influences prevent this near the center of the field. A further basic attribute of this configuration, to which attention is given in the following, is readily indicated if small foreign particles are floated on the water surface, after assuming the natural flow pattern, and their apparent efforts to retain a constant angular orientation are observed.

A formulation of this situation is that

$$p/\rho + gz + u_t^2/2 = \text{constant} \tag{9-5}$$

or, differentiating with respect to radial position in the flow-field,

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\right) + g\left(\frac{\partial z}{\partial r}\right) + u_t\left(\frac{\partial u_t}{\partial r}\right) = 0$$

and $$\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\right) + g\left(\frac{\partial z}{\partial r}\right) = -u_t\left(\frac{\partial u_t}{\partial r}\right)$$

Comparing this particularized relation with the general one of equation 9-4

$$-u_t\left(\partial u_t/\partial r\right) = u_t^2/r \quad \text{or} \quad \frac{\partial u_t}{\partial r} + \frac{u_t}{r} = 0 \tag{9-6}$$

A further unique feature of the free vortex is provided by writing Eq. 9-6 in the form

$$u_t\,\partial r + r\,\partial u_t = \partial(u_t r) = 0$$

whereby $$u_t r, \text{ or } \omega r^2 = \text{constant} \tag{9-7}$$

The constancy of this product, $u_t\ r$ or $\omega\, r^2$, makes it an index characterizing the motions existing <u>throughout</u> a given free-vortex field. It has been referred to as a measure of the "strength" of the vortex; the product $2\pi u_t r$ becomes significant in subsequent mathematical analysis of other flow patterns which are related to that of the free vortex, and is known as the <u>circulation</u> (Γ).

A further capability of this constant product lies in its provision of the requisite further information enabling particularization and integration of the general relation of Eq. 9-4. By introducing Eq. 9-7 in the free-vortex energy relation of eq. 9-5, for that configuration

$$(p/\rho + gz) - (p_1/\rho + gz_1) = (u_{t,1}^2 - u_t^2)/2 \quad \text{and, as } u_{t,1} = \frac{u_t r}{r_1},$$
$$= \frac{(u_t r)^2}{2}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right) \tag{9-8}$$

The several criteria which are so provided, for describing the free-vortex flow configuration, require radial variations of u_t, $u_t^2/2$, u_t/r and $\frac{\partial u_t}{\partial r}$ (eq. 9-6), $p/\rho + gz$, and the total energy per unit mass, such as are illustrated in the elevation view of Fig. 9-3. In the plan view the separations between those stream lines shown by arrows are such (Art. 5-2e, and Fig. 5-1) that they represent the boundaries of stream tubes through which there are <u>equal volume-rates of flow</u>. The separations indicate also, but inversely, the mean velocity in the tubes. Like representations are made in various of the following figures. Quantities in the accompanying table are ones corresponding to an adopted value of 2.5 for $u_t r$, or 6.25 for $(u_t r)^2$, r_1 selected as 0.50, and independently imposed conditions such that $p_1/\rho + gz_1 = 100$.

r	$1/r^2$	$\frac{1}{r_1^2} - \frac{1}{r^2}$
1.25	0.64	3.360
1.00	1.00	3.000
0.75	1.778	2.222
.50 = r_1	4.0	0.0
0.25	16.0	- 12
0.10	100	0 96

$u_t r$	u_t	$u_t^2/2$	$u_t/r = \partial u/\partial r$
2.5	2.00	2.0	1.60
"	2.50	3.125	2.50
"	3.333	5.556	4.44
"	5.0 = u_{t1}	12.5	10.0
"	10.0	50.0	40
"	25.0	312.5	250

$(p/\rho+gz) - (p_1/\rho+gz_1)$	$p/\rho + gz$	$p/\rho + gz + u^2/2$
10.52	110.5	112.5
9.38	109.4	"
6.94	106.9	"
0.00	100.0	"
-37.5	62.5	"
- 300	(-200)	"

The solid line in the figure exhibits the typical form of the free surface of a liquid vortex when the surface is exposed to constant (atmospheric) pressure and motion is about a vertical axis. But an exception is to be noted, near the center of rotation, where the above criteria are incapable of realization with an actual fluid.

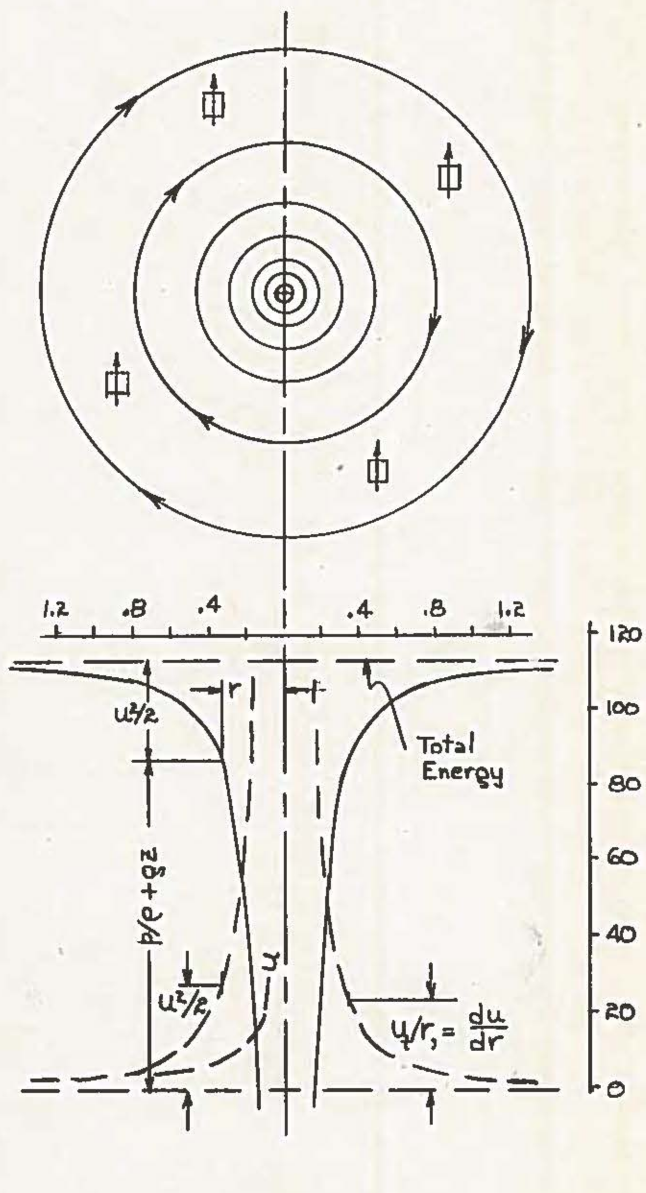

Figure 9-3

That is, for any finite value of $u_t r$ the values of u_t (or ω) and of the velocity gradient ($\partial u_t/\partial r$) would need approach infinity as r approaches zero, and $p/\rho + gz$ approach minus infinity. These are absurdities. Also the viscosity of a real fluid causes the flow pattern near the center to approach that of the viscous vortex (art. 9-6).

A distinctive feature of the free vortex flow pattern appears if Eq. 9.6 $\left(-\partial u_t/\partial r = u_t/r\right)$ is interpreted by reference to Fig. 9-4. The ratio $u_t/r\ (=\omega)$ evidently expresses the (counterclock-wise) angular velocity of the tangential bisector of the particle about its centroid. The radial bisector also rotates about the centroid, but in a clockwise sense and with magnitude $\dfrac{u_t + \frac{\partial u_t}{\partial r}dr - u_t}{dr} = \dfrac{\partial u_t}{\partial r}$. Since by Eq. 9-6 these two angular velocities are equal in magnitude but opposite in sense, the particle moves and deforms in such a manner that it retains a constant angular orientation in space. This is indicated in Fig. 9-3. Moreover, since the "net" angular velocity of the particle about its centroid is zero, the particle is said to have no rotation, or vorticity and the motion is said to be irrotational. As the condition of irrotation is a direct consequence of frictionless flow and uniform energy distribution, this type of motion can not be obtained with real fluids where viscosity influences exist. However, the considerations above find great utility in describing a situation which may be ideally approached. The hydrodynamic concept of an irrotational flow bears striking similarity to the thermodynamic concept of an isentropic process.

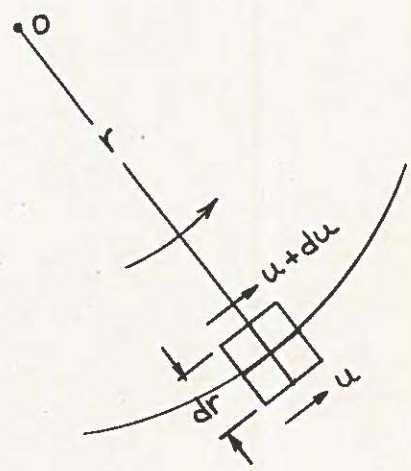

Figure 9-4

Paradoxically, although the irrotational flow is ideal in the sense of being frictionless, by reason of the equi-energy distribution it has in fact no utility in practical situations where it is desired to change the energy level of the fluid.

9-6. Forced (Circular) Vortex. In contrast to the free-vortex type of circulatory motion, where ωr^2 is constant, one may readily picture the opposite situation in which a uniform angular velocity ω is enforced throughout a rotating flow field. Such a configuration is known as a <u>forced vortex</u> and may be approached, for example, by a mass of fluid in a circular vessel if caused to move by a concentric rotor with a large number of radial blades. The fluid and its constituent particles rotate in the hypothetic forced vortex, quite as if it were a solid body (Fig. 9-5a).

An essential feature of this flow configuration is seen by introducing in the general relation of Eq. 9-4 the specification that ω is constant, whereby that relation becomes

$$dp/\rho + g\,dz = \omega^2 r\,dr$$

With ω constant and if ρ changes negligibly this may be integrated to

$$(p/\rho + gz) - (p_1/\rho + gz_1)$$
$$= \omega^2 (r^2 - r_1^2)/2$$
$$= \omega^2 (r^2 - r_1^2) - \frac{\omega^2}{2}(r^2 - r_1^2)$$

or, as $\omega = u/r$, and is constant, may be modified to the form

$$(p/\rho + gz + u^2/2) - (p_1/\rho + gz_1 + u_1^2/2)$$
$$= \omega^2 (r^2 - r_1^2) \qquad (9\text{-}9)$$

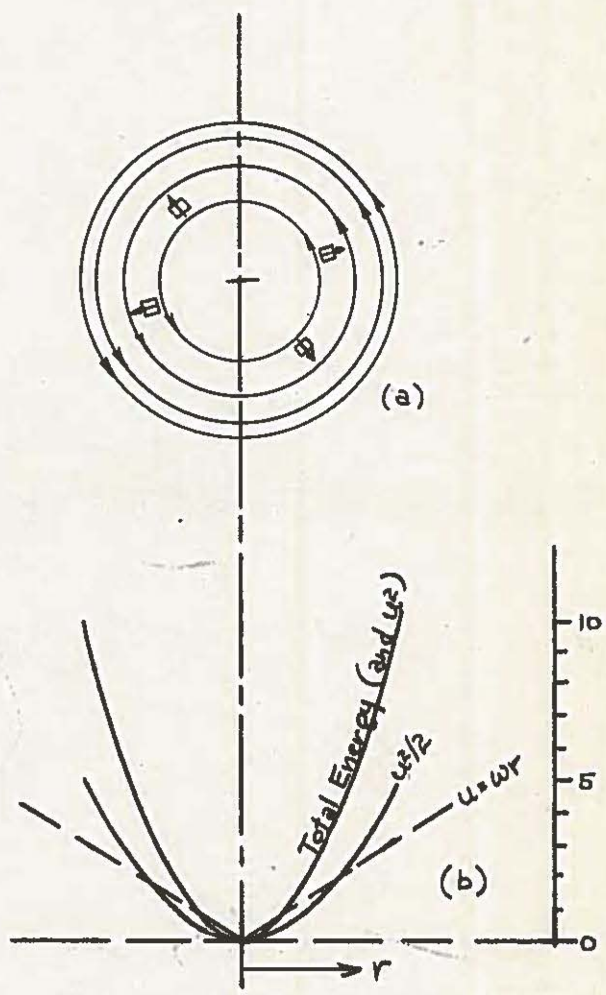

Fig. 9-5 Forced Vortex

At a given angular velocity for all, the aggregate mechanical energy attributable to any individual particle at radius r thus exceeds that at r_1 by the difference between their squares. A corresponding radial variation of u, $u^2/2$, $p/\rho + gz$ and the aggregate energy per unit mass is exhibited in Fig. 9-5, and is evidently characterized by active energy increase at increasing radial distances. This contrasts with the equi-distribution of the free vortex.* Complete attainment of the forced-vortex flow pattern is opposed and prevented by the invariable tendency toward the equi-distribution of energy typifying the free vortex. For example, in the rotor with radial blades this operates to induce a secondary "circulation" within the cell between any adjacent pair of blades. However attention to the forced vortex is justified as its characteristics are highly desirable and are more or less closely approached in the rotor of engineering equipment such as the centrifugal pump, radial-flow hydraulic turbine, hydraulic coupling or torque-converter.

9-7. Viscous Vortex. As indicated by its designation, in this configuration the relative velocities throughout the fluid, and in any cylindrical lamina therein, are both generated and governed by the shearing resistances due to the viscosity of the fluid. They are generated by an enforced rotation of a cylindrical rotor either (a) immersed in a fluid field or (b) enclosing it, but with a concentric cylindrical stator occupying the conjugate position. The diagrams of Fig. 9-6 illustrate the arrangement. At the surfaces the velocity of the adhering fluid is the same as that of the surface.

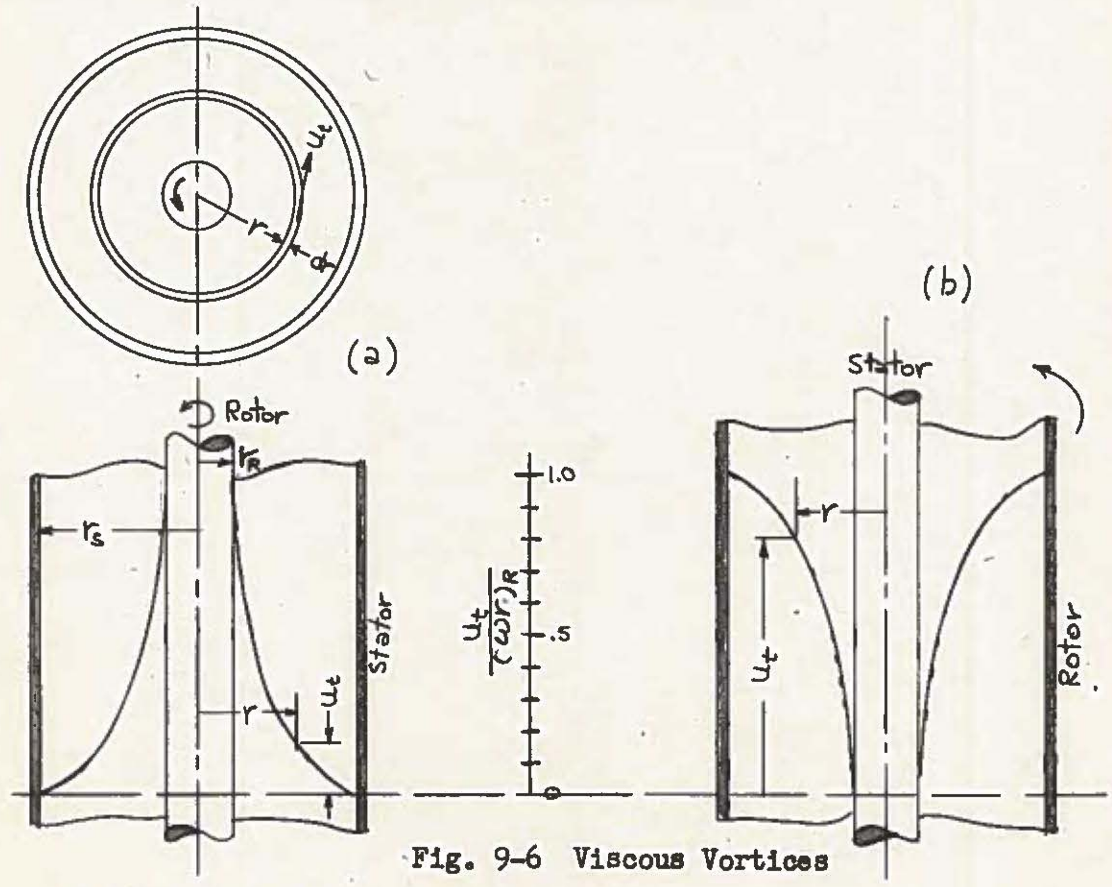

Fig. 9-6 Viscous Vortices

* See footnote p 172

Referring to the lamina in the plan view of sketch (a), of radius r and thickness dr and of length L, from the definition of viscosity (art. 1-8) the (tangential) force required for producing the relative motion may be expressed as

$$F_t = -\mu A\,(du/dr), \text{ or } F_t/(2\pi r L\mu) = -du/dr$$

also

$$\frac{F_t r}{2\pi L\mu}\,\frac{dr}{r^2} = -du$$

But the product $F_t r$ is recognized as the torque (q) required for maintaining the relative motion, and also is necessarily constant throughout, else angular accelerations would develop in a field in which the motions are by specification steady.

Correspondingly the last relation may be put in the integrable form

$$\frac{q}{2\pi L\mu}\int_{r_R}^{r} r^{-2}dr = -\int_{u_R}^{u} du$$

$$\frac{q}{2\pi L\mu}\int_{r_R}^{r} d(r^{-1}) = u - (\omega r)_R \qquad (9\text{-}10)$$

where subscript R refers to the rotor, and in the following subscript S will refer similarly to the stator. Also

$$u = (\omega r)_R + \frac{q}{2\pi L\mu}\int_{r_R}^{r} d(r^{-1}) \qquad (9\text{-}11)$$

For the arrangements of Fig. 9-6 the boundary conditions (or integration limits) are such that the results of integration of these relations take the following forms.

(a) With stator surrounding rotor; at $r = r_s$, u = zero, so that

$$\frac{q}{2\pi L\mu} = \frac{\omega_R r_R}{\left(\frac{1}{r_R} - \frac{1}{r_s}\right)} \qquad (9\text{-}12)$$

also

$$u = (\omega r)_R + \frac{q}{2\pi L\mu}\int_{r_R}^{r} d(r^{-1})$$

$$= (\omega r)_R\left[1 - \frac{(1/r_R - 1/r)}{(1/r_R - 1/r_s)}\right]$$

$$= (\omega r)_R\left(\frac{1/r - 1/r_s}{1/r_R - 1/r_S}\right) \qquad (9\text{-}13)$$

* - A further contrast relates to the rotation, or vorticity, in the two flow configurations; i.e., the relative magnitudes of the index defined as the sum $(\partial u_t/\partial r + u_t/r)$ of Eq. 9-6. In the free vortex this was seen to equal zero; in the forced vortex both u_t/r and $\partial u_t/\partial r$ equal ω, and ω is constant. Thus

$$\partial u_t/\partial r + u_t/r = 2\omega \qquad (9\text{-}9a)$$

or the rotation or vorticity equals 2ω.

A velocity distribution such as represented by this relation is indicated in the elevation-view of Fig. 9-6.

Instruments utilizing these considerations are employed for determination of the (absolute) viscosity of fluids, through measurements of the torque required to maintain a given angular velocity of rotor, but with due precautions such as accounting for end effects and assurance that velocities and velocity gradients are sufficiently low that fluids motions remain laminar.

(b) With rotor surrounding stator, the relations are the same, but with some convenience in writing the velocity relation as

$$u = (\omega r)_R \left(\frac{1 - r_S/r}{1 - r_S/r_R} \right)$$

In the absence of a stator, the turning of the rotor and the viscous resistance to relative motion between adjacent laminas of the fluid it surrounds, cause the fluid mass ultimately to assume the flow pattern of the forced vortex, except as modified by the trend to equi-distribution of the mechanical- energies aggregate.* On abrupt stoppage of the rotor there is immediate evidence of efforts to assume the free vortex pattern and in fact a prompt assumption of it to such degree as is possible with an actual fluid. The situation parallels that noted in the footnote of art. 9-4.

9-8. Pipe Bend or Elbow as Flow Meter. The free-vortex velocity and pressure distributions are sufficiently closely approached at the mid-section of a well formed elbow or pipe bend, if producing some 90° total stream deviation, that measurement of the pressure difference as between radially inner and outer surfaces of the channel may be utilized for flow-rate metering. Fig. 9-7 illustrates, again suggesting the velocity distribution by stream-line separations.

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* - This is typical of the conditions approached at the "eye" of a violent cyclonic disturbance in the atmosphere, in which a calm is encountered at the center and velocities increase to a maximum at some distance therefrom, but outside of that they tend to exhibit the radial distribution of the free vortex. The energy maintaining the central viscous-forced vortex is provided by that of the enveloping fluid.

For the aggregate of the stream tubes forming the stream, of individual breadths dr,

$$\dot{V} = \int u_t \; dA, \text{ and for free-vortex configuration,}$$

$$= (u_t \; r) \int dA/r, \text{ and by Eq. 9-8a}$$

$$= \sqrt{2\left[(p_2/\rho - gz_2) - (p_1/\rho + gz_1)\right]} \; \frac{\int_{r_1}^{r_2} dA/r}{\left[1/r_1 - 1/r_2\right]^{1/2}} \tag{9-14}$$

$$\dot{V} = A\sqrt{2\left[(p_2/\rho + gz_2) - (p_1/\rho + gz_1)\right.} \;\; f(r_2/r_1) \tag{9-14a}$$

where A = cross-sectional area of stream. Integration of the

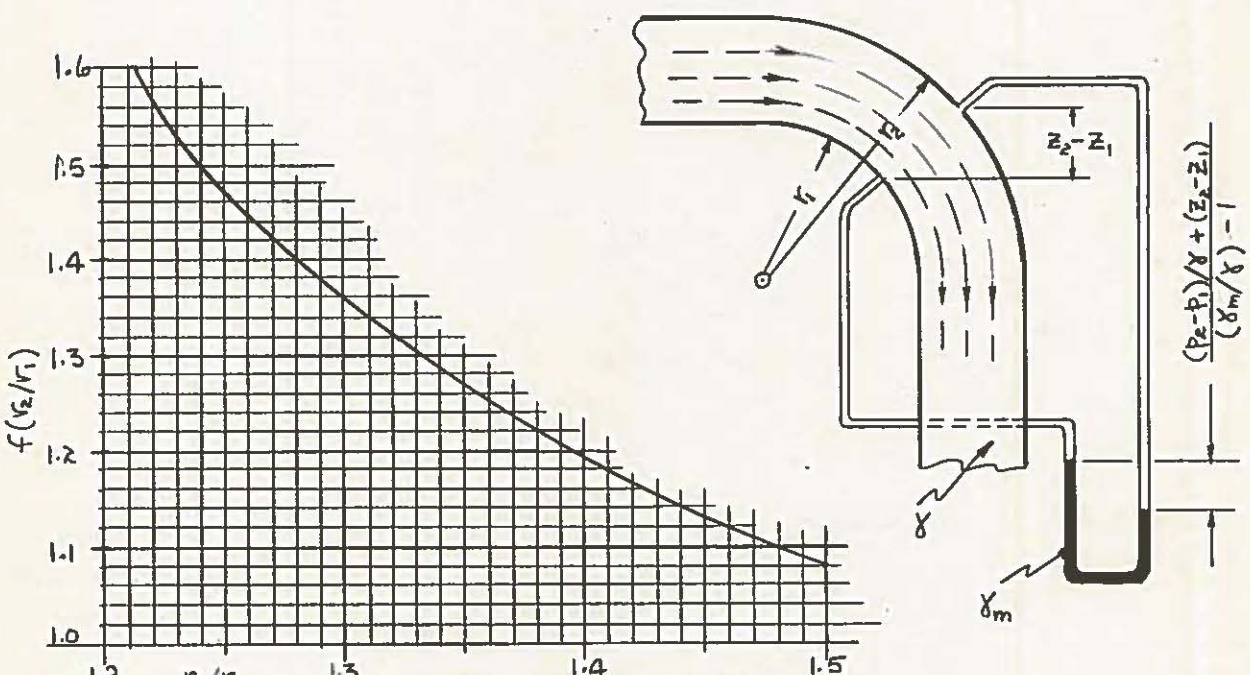

Fig. 9-7 Elbow Meter

indicated function of r_2/r_1, both for the effectively two-dimensional flow in a rectangular channel and three-dimensional in a circular one, give values for either which differ trivially within the limits indicated by the accompanying curve.

Calibration tests have indicated that ratios of actual to anticipated flow-rates, or "discharge coefficients" based on anticipated rates as determined by Eq. 9-14a and the curve, are quite closely unity (1.0) if $\dot{V}/A > 2$ ft/sec, $r_2/r_1 > 1.5$ and the stream deflection to the pressure-measurement section is some 45°. Pressure differences are relatively small, requiring precision both in their measurement and the radius-ratio data.

9-9. Spiral and Helical Vortices. The purpose of many machines is to provide for the impartation of mechanical energy to, or its departure from, a fluid stream while that is also being caused to pass steadily through the machine. In more modern machines these objectives are accomplished by the combining, or <u>superimposing</u>, of several individual types of flow pattern. The centrifugal pump and hydraulic turbine illustrate.

Simpler ones of such synthesized flow configurations are the result of superimposing a steady <u>radial</u> flow on a free or a forced vortex, the resulting flow patterns being known as free, or forced, <u>spiral vortices</u>. In these the radial velocity component will evidently be such that

$$u_r = \frac{\dot{V}}{2\pi y r}$$

where y is the thickness of the flow sheet and $2\pi r y$ its consequent transverse area. It is sufficient for present purposes to limit attention to a sheet of constant thickness, and thus to a steady two-dimensional flow in which the radial velocity of any particle in the stream varies inversely with its radial distance from origin 0; also $\frac{du_r}{dr} = -\frac{\dot{V}}{2\pi y r^2} = -\frac{u_r}{r}$

The consequent radial acceleration of the particle is such that

$$a_r = du_r/dt \quad \text{or} \quad -(du_r/dr)(dr/dt), \text{ and as } dr/dt = u_r$$
$$= -u_r(du_r/dr) \quad \text{or} \quad -\tfrac{1}{2}d(u_r^2)/dr$$

The superimposed radial flow thus involves the action of a supplemental radially-directed force on the particle of such character that

$$F_r = m a_r = -\rho(dr\,dy\,ds)\frac{1}{2}\frac{d(u_r^2)}{dr}$$

or dividing by $-\rho(dr\,dy\,ds)$,

$$\frac{F_r}{\rho(dr\,dy\,ds)} = \frac{1}{2}\frac{d(u_r^2)}{dr}$$

Correspondingly supplementing the equilibrium relation of Eq. 9-4, relating to the vortex alone, but recognizing that the partial derivatives of that relation may in this case be taken as total derivatives,*

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* - A more rigorous derivation of a relation expressing the total radial acceleration is provided by writing the total derivative du_r as $du_r = (\partial u_r/\partial t)dt + (\partial u_r/\partial s)ds + (\partial u_r/\partial r)dr$

Thus $a_r = du_r/dt = (\partial u_r/\partial t) + (\partial u_r/\partial s)(ds/dt) + (\partial u_r/\partial r)(dr/dt)$

or, as (Fig.9-1) $ds/dt = u_t$ and $dr/dt = u_r$

$$a_r = (\partial u_r/\partial t) + u_t(\partial u_r/\partial s) + u_r(\partial u_r/\partial r)$$

But for steady or time-independent flow $(\partial u_r/\partial t) = 0$ and for an angular independent flow pattern by eq.9-2 $(\partial u_r/\partial s) = u_t/r$

and $a_r = u_t^2/r + u_r(du_r/dr)$

Here the first term to the right is recognized as the "centripetal" acceleration of the footnote of art. 9-2, associated with the circular path of the vortex flow-component, and the second as the above acceleration accompanying change in radial position.

$$\frac{1}{\rho}\frac{dp}{dr} + g\frac{dz}{dr} - \frac{u_t^2}{r} + \frac{1}{2}\frac{d(u_r^2)}{dr} = 0$$

or
$$\frac{dp}{\rho} + g\,dz = \frac{u_t^2}{r}dr - \frac{1}{2}d(u_r^2) \qquad (9\text{-}14a)$$

But also, as $u_r^2 = u^2 - u_t^2$ where u = resultant of the tangential velocity component of the simple vortex motion and the radial component of the superimposed radial flow,

$$\frac{dp}{\rho} + g\,dz + \frac{1}{2}d(u^2) = \frac{u_t^2}{r}dr + \frac{1}{2}d(u_t^2) \qquad (9\text{-}14b)$$

This provides a general relation which may be adapted as follows to the particular conditions of the two-dimensional free or forced spiral-vortex flow patterns.

(a) Free Spiral Vortex. Recalling that for the free vortex (eq. 9-7) $u_t r$ = constant, or $u_t^2 = C^2/r^2$, and also noting that the terms to the right in Eq. 9-14b may thus be written as $\frac{u_t^2}{r}dr = C^2r^{-3}dr$ and as $\frac{1}{2}d(u_t^2) = -C^2r^{-3}dr$ by introducing these particular conditions in the general relation $dp/\rho + g\,dz + \frac{1}{2}d(u^2) = 0$

$$p/\rho + gz + u^2/2 = \text{constant} \qquad (9\text{-}15)$$

That is, as this aggregate represents the total mechanical energies attributable to the particle, the ideal free spiral vortex is in fact <u>not</u> capable of providing for the mechanical energy transition between particle and environs which was suggested above as an objective of many machines. But this need not be disappointing, as the effecting of a change in the <u>distributions</u> of the individual energy items, as between flow work and pressure, geopotential and kinetic energy, and as the flow procedes, is not precluded and becomes quite necessary in portions of such machines. The influence of fluid viscosity and friction prevent full attainment of the ideal flow configuration.

A feature of the two-dimensional free spiral vortex is seen by noting that, from the initial relation above and from Eq. 9-7, both $u_t\,r$ and $u_r\,r$ are constant. Thus at a given value of r and at all points along a circle of that radius, $\tan^{-1}(u_r/u_t)$ and thus the orientation of all stream lines is the same. The resulting identical (spiral) character of all stream lines throughout the stream is indicated in Fig. 9-8.

(b) Forced Spiral Vortex. Although one may not associate kinematic aspects of the flow pattern assumed by a stream when radial flow is taken to accompany the rotational motions of the forced vortex,* informative energy accountings may be provided indirectly if (vectorial) force relations are transformed to (scalar) energy relations, such as equation 9-14a.

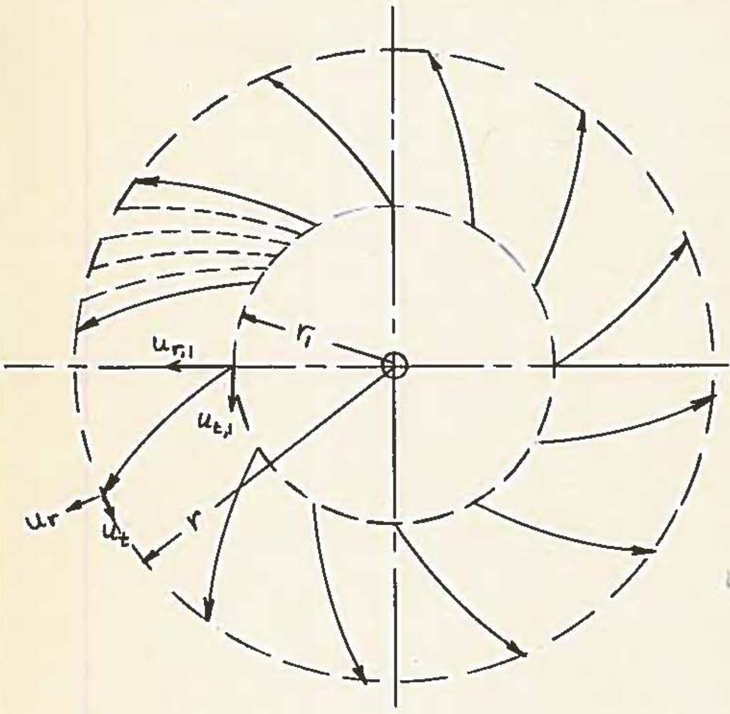

Fig. 9-8 Free Spiral Vortex

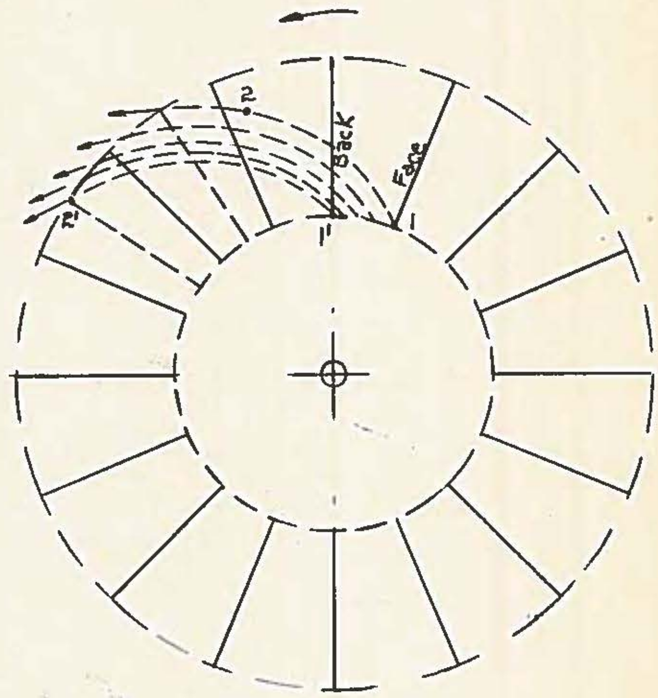

Fig. 9-9 Forced Spiral Vortex

So proceeding, let the specification of constant u_t/r associated with steady radial flow in a stream line be introduced im the general energy relation of Eq. 9-14a. Consequently in that relation $\frac{u_t^2}{r}dr = \omega^2 r dr = \omega^2 \frac{d(r^2)}{2}$.

Thus $$\frac{dp}{\rho} + g dz = \frac{\omega^2}{2} d(r^2) - \frac{1}{2} d(u_r^2) \quad ;$$

Writing this in terms of definite integrals,

$$\left(\frac{p}{\rho} - \frac{p_1}{\rho_1}\right) + g(z - z_1) = \frac{\omega^2}{2}(r^2 - r_1^2) - \frac{1}{2}(u_r^2 - u_{1,r}^2)$$
$$= \frac{1}{2}(u_t^2 - u_{1,t}^2) - \frac{1}{2}(u_r^2 - u_{1,r}^2)$$

But $$\frac{u^2}{2} = \frac{u_t^2}{2} + \frac{u_r^2}{2} \quad \text{so that}$$

$$\left(\frac{p}{\rho} - \frac{p_1}{\rho_1}\right) + g(z - z_1) + \frac{1}{2}(u^2 - u_1^2) = u_t^2 - u_{1,t}^2$$
$$\text{or} \quad = \omega^2 (r^2 - r_1^2) \qquad (9\text{-}16b)$$

The interpretation of the evidence furnished by this relation, as applied to particles located along any radial line, is

(a) that this radial-flow and forced-vortex combination accomplishes the objective of energy impartation to, or departure from, the stream as the particles

* See Footnote page 178

proceed through that flow configuration, in contrast with the situation with the free spiral vortex; and

(b) that the energy so transferred per unit mass of fluid will be a function of the square of the angular velocity of the particles, and the squares of the radial distances at entry to and departure from the flow field.

Such evidence is supported, for example, by the ability of the conventional "centrifugal" pump, which provides a flow configuration of this general nature, to deliver a fluid steadily against pressures and at velocities much exceeding those at entry; also by test evidence on geometrically similar pumps, operated under dynamically similar conditions, indicating that the energy imparted is in fact proportional to the square of the rotative speed of its rotor and to that of its diameter.

In further contrast to the flow pattern of the free spiral vortex, the resultant velocities in contiguous stream lines and tubes may now not be the same at all angular positions along a circle of given radius (relative to 0) in the flow field. The "mechanism" whereby energy impartation (or departure) is accomplished requires instead that, in each cell of a rotor which is endeavoring to enforce the flow configuration, there be a progressive variation of the resultant velocities, and associated pressures, along a circle of given radius.

Analyses indicating the exact manner of such variation are difficult, if not impossible. However, although a radially-bladed rotor is rarely used, present purposes will be served by reference to the representation of such a rotor in Fig. 9-9, regarded as a pump rotor. Its rotation must be enforced by the action of a torque, furnished by engine or motor, but also resisted by

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* - That is, methods of superposition which will subsequently be employed, in the analyses of Part V, are valid and of much advantage for associating kinematic aspects of simultaneous but different forms of irrotational flow. However, they become invalid for this purpose when, for example, the rotational character of motions of the forced vortex, Arts. 9-4 and 9-5, accompany the inherently irrotational flow characterizing purely radial flow.

an equal opposing torque, else acceleration of the rotor would occur.

The energy impartation to (or departure from) the stream while en route through the rotor evidently necessitates an accompanying energy transition from (or to) the rotor, but further a concurrent energy passage to (or from) the rotor via its shaft. For convenience considering particularly a pump rotor, its rotation must be enforced by the action of a torque furnished by engine or motor, but also be resisted by an equal opposing torque, also acceleration of the rotor would occur. The source of this opposing torque can only be an excess of the pressures on the advancing side or "face" of a blade forming one boundary of a cell of the rotor, relative to the pressures on the retreating side or "back" of the blade forming the other boundary. This provides the "mechanism" whereby the energy impartation to the stream is accomplished. Although a radially-bladed rotor is rarely used, present purposes will be served by reference to a representation of such a rotor in Fig. 9-9.

Within each rotor cell, the distribution of velocity is related to the distribution of pressure. In conformity with the fluid law of Art. 9-2, the transverse energy distribution tends to become uniform. Consequently, along the face of the blade where the pressure is relatively high, the velocity will be less than that along the back of the blade.

Utilizing the technique of Fig. 5-1 (Art. 5-2), such a velocity distribution is indicated in the figure by the relative separations of stream lines bounding adjoining stream tubes through which equal flow rates ($\dot{V}$) are occurring in the aggregate stream passing through the moving cell. It also indicates, indirectly, the converse pressure (and flow-work) distribution necessary for the energy transition between blades and stream. The heavier solid-line and broken-line representations of a cell indicate respectively an initial and a subsequent position, after advancing about center 0 in a given time interval. Stream line 1-2 denotes the travel of a particle at lesser velocity along the advancing face of a blade; and is in contrast to the greater travel of a particle, in the same time, along the retreating back of the adjoining blade and from an initial point 1' to a second point 2'.

If any type of two-dimensional vortex is superimposed on a uniform flow in the third, or axial direction, each streamline of the combined flow is a helix and the flow is known as a helical vortex. It is frequently encountered in hydraulic equipment in at least two common types of situation. The first, as encountered for example, in the draft tube of a hydraulic turbine, is essentially the superposition of a free vortex flow on a uniform axial flow. The second is one in which an enforced angular acceleration is caused to accompany an axially directed flow, and is characteristic of the flow through a ship's propeller, or like devices employed in distinctive types of pumps and turbines. Exact analyses of the flow under the latter conditions become quite complex, but is given some attention in subsequent considerations of specific equipment.

9-10. The "Wall Force". In the several preceding articles, and the closing portion of the last, there were indicated (a) the nature and inter-relations of the forces and restraints determining the motions of the particles forming a stream line, and the bundles of stream lines forming a stream tube or an entire steam; and (b) the "mechanism" whereby reactive forces exerted by the walls confining a stream may contribute to the generation of the forces on the particles. In art. 5-12, relating to the momentum equations and by a more macroscopic approach in which the myriad of particles forming a finite segment of a stream was regarded as the equivalent of the free body of solid-body mechanics, a convenient "short-cut" but admittedly approximate method was noted whereby one might express, in terms only of the mass-rate of flow of a stream and any difference in (mean) terminal velocites, the single resultant force required for effecting that velocity change.

In a parallel manner, and tolerating like approximations, a very convenient engineering tool may be provided which enables evaluation, solely from information on the (mean) pressures and velocities at end-sections of a stream segment, of the aggregate force exerted on it by the walls of the channel confining it and compelling any change in its state of motion; or, conversely, those exerted on the walls by virtue of the passage of the stream. The second viewpoint is frequently of more direct engineering concern. With the background provided in foregoing material, it will be sufficient here to illustrate initially by attention to flow in which the velocity change is one only of magnitude, and to a stationary channel. Chapter 10 will give attention to applications with machines in which the force acts on moving parts, and enables desired energy transfer as work.

Referring to Fig. 9-9 and the transverse elemental segment of a fluid stream, of density $\underline{\rho}$ and of thickness $\underline{ds}$ equal to the distance it will have advanced in time $\underline{dt}$ in the indicated channel, the several forces influencing any change in the state of motion are as follow . Motions or forces in the direction of advance will be regarded as positive in sense.

(1) The inertial restraint opposing any acceleration, if the mean velocity of advance of the segment may be regarded as representable by a single vector, is seen in Art. 5-12 to be expressible as $-\dot{m}\, du_s$.

(2) The net force due to the pressures at upstream and down-steam faces of the segment is in amount $pA - (p + dp)(A + dA)$ or $- A\, dp - p\, dA$, and thus $-d(pA)$; where A is <u>normal</u> to the direction of advance.

(3) The s-component of the gravitational force, of magnitude and direction $- g \rho A\, ds \sin \phi$; but with $\rho g A\, ds$ being the weight of the segment and the foregoing thus being expressible as δ wt $\sin \theta$.

(4) The force exercised on the segment by the surface confining it, of amount $p\, dA$ and with positive or negative significance attaching to annular area dA as the channel area respectively increases or decreases on proceeding in the direction of advance.

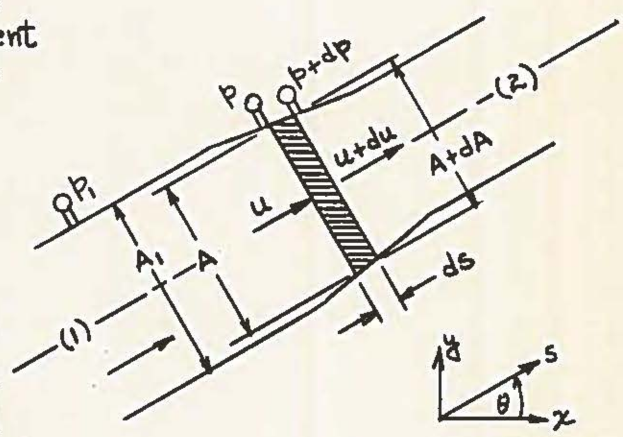


Figure 9-10

(5) The s-component of any retarding frictional resistance, atributable to fluid viscosity and represented by symbol $- \delta F_f$.

An equilibrium equation associating these forces would be

$$-\dot{m}\, du_s - d(pA)_s - \delta wt \sin\theta + (p\,dA - \delta F_f) = 0$$

But for most purposes it is sufficient (and more convenient, and so is customary) to account independently for the weight of the fluid in a conduit, and thus to limit attention to such pressures and pressure differences as are attributable to inertial and frictional influences associated with the flow. With such interpretation of the pressure items, and also rearranging the above relation and writing it in integrated form, it be-

becomes $\qquad -\int_1 p\,dA + \Sigma F_f = -\int_1 d(pA) - \dot{m}\int_1 du$

Observe here that the terms to the left provide a summation of those forces, described in the above sub-paragraphs (4) and (5), which are specifically contributed by the surfaces confining the fluid stream while en route through the channel and experiencing consequent change in its state of motion. In opposite direction sense, it is that acting <u>on the channel walls</u> through the agency of the moving stream and by reason of its flow. In this sense the item becomes one of major concern in analyses of the (dynamic) forces associated with flow through confining channels, and as such is suitably referred to as the <u>wall force</u>, F_W, acting on the channel.

Direct evaluation of the individual terms of this summation, through information on the progressive variations of pressure and area and on the frictional influences, may well be, however, at least quite bothersome and in many instances impracticable. But escape from this difficulty is seen to be provided by noting that the terms to the right in the above relation, or $-\int_1 d(pA)$ and $-\dot{m}\int_1 du$, are directly integrable to the form $(p_1 A_1 - pA) + \dot{m}(u_1 - u)$, and so represent differences between terminal magnitudes which are independent of interim variations of p, A or u. That is, to the extent that suitable (mean) magnitudes may be ascribed to terminal values of p and u, the wall force may be evaluated readily but indirectly through the relation

s-component of wall force, $F_{W,s} = (p_1 A_1 - pA)_s + \dot{m}(u_1 - u)_s$ (9-17)

Although in this relation the terms in <u>pA</u> and <u>$\dot{m}u$</u> may conveniently be utilized and loosely be referred to as "pressure forces" and "velocity forces", it is to be emphasized that they may not be regarded as individual forces acting at entry and exit sections of the channel. Instead, their differences merely provide an indirect evaluation of an aggregate of forces which are <u>distributed</u> along the interior surfaces of the channel. The following example utilizes the relation.

<u>Example 9-1</u>. Fresh water having a density of 1.94 slugs (62.4 lbm) per cu.ft is being delivered through the horizontal nozzle of Fig. 9-11 at the rate of 3.55 cu.ft/sec. Determine the wall force on the nozzle, and its direction.

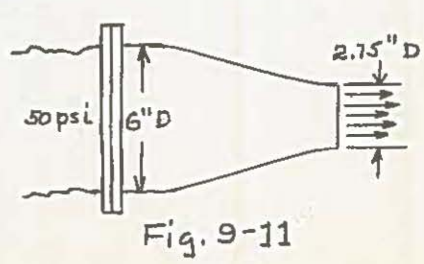


Fig. 9-11

Solution. General items:- $\dot{m} = 3.55 \times 1.94 = 6.887$ slugs (221.6 lbm) per sec.; $A_{pipe} = (\pi/4)(6/12)^2 = 0.1964$ sq.ft.; $A_{jet} = 0.0413$ sq.ft.; $\bar{u}_{pipe} = 18.08$ ft/sec; $\bar{u}_{jet} = 86.06$ ft/sec.; $p_1A_1 = (50 \times 144) \times .1964 = 1414$ lbf.; $pA = 0.0$. (The use of gage pressures is permissible as the atmospheric pressure is acting through out the exterior of the assembly.) Thus,

$$F_W = 1414 + 6.887 \times (18.08 - 86.06) = +946 \text{ lbf}$$

This indicates that the nozzle is so acted upon by the stream as to tend to move it in the direction of the flow; or, for restraining such motion, a force of 946 lbf must in some manner be applied in a direction opposite to that of the flow.

For flow situationsinvolving change in direction as well as ones of pressure and speed, equation 9-17 is directly adaptable if force components in each coordinate direction are individually accounted. That is,

$$F_{W,x} = \left[(p_1A_1)_x - (pA)_x\right] + \dot{m}(u_{1,x} - u_x), \text{ and}$$

$$F_{W,y} = \left[(p_1A_1)_y - (pA)_y\right] + \dot{m}(u_{1,y} - u_y). \qquad (9\text{-}17a)$$

The following example illustrates the use of these relations.

Example 9-2. The situation of example 9-1 is supplemented by the additional piping of Fig. 9-12. The pressures at the indicated points decrease progressively due to the frictional and turbulence influences between them. Regarding the plane of the assembly as horizontal, determine (1) For the assembly as a unit, the the x and y components of the wall force, and the net moment about 0; (2) The x and y components of the wall forces on individual parts a, b and c of the assembly, checking also the aggregate of these against the results of part 1 and interpreting the several x and y components.

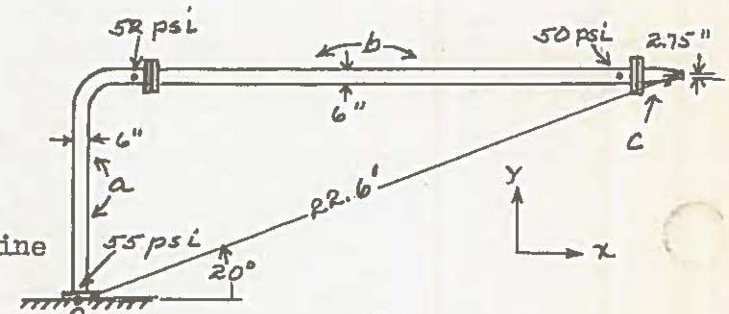


Fig. 9-12

Solution. General items are as in example 1.

(1) For the assembly; $F_{W,x} = (0 - 0) + 6.887(0 - 86.06) = -593$ lbf,

$$F_{W,y} = (55 \times 144) \times .1964 - 0 + 6.887(18.08 - 0) = +1680 \text{ lbf}$$

By eq. 5-9, moment about $0 = 6.887(86.06 \times \sin 20^\circ \times 22.6 - \text{zero}) = 4580$ lbf.ft.

(2) For the individual components:

		$F_{W,x}$	$F_{W,y}$
Through bend	$F_{W,x} = [0 - (52 \times 144) \times .1964] + 6.887(0 - 18.08) =$	-1596	
	$F_{W,y} = [(55 \times 144) \times .1964 - 0] + 6.887(18.08 - 0) =$		1680
Bend to nozzle	$F_{W,x} = (2 \times 144) \times .1964 + 6.887(18.08 - 18.08) =$	57	
	$F_{W,y} =$ zero		0
Nozzle	$F_{W,x} =$ (from example 9-1)	946	
	$F_{W,y} =$ zero		0
	Resultant	-593	1680

The resultants agree with those computed directed for the assembly. For the bend the force to the left (-1596 lbf) is the total of the "pressure-and-velocity forces" due to the diversion of the stream through 90°, but this is ~~partially~~ partially compensated by the rightward frictional drag of 57 lbf in the pipe and the 946 lbf causing tension in the flange joining the nozzle to the pipe. The net leftward force of 693 lbf produces shear in the bend fastenings at the bulkhead, and also the above bending moment about 0. The 1680 lb force is a tensile loading at the bulkhead fastenings.

The 4580 lbf.ft moment about 0 acts to so distribute the above tensile loading of 1680 lbf that a greater proportion of it acts to the right. Incidentally, such a moment is the source of the "flailing-about" tendency exhibited by an unrestrained nozzle at the end of a flexible hose if the hose is not straightened out, or would be the agency causing the assembly to rotate about 0 if permitted to do so, as in the familiar lawn sprinkler.

A non-dimensional arrangement of the wall-force relation may be convenient, particularly if the fluid density varies. By referring to eq. 9-17, noting that $\dot{m} = (\rho u A)_s$ and suitably collecting the terms of that equation,

$$\left(\frac{\bar{F}_W}{\dot{m}\, u_1}\right)_s = \left(\frac{p_1/\rho}{u_1^{\,2}} + 1\right)_s - \frac{u_s}{u_1}\left(\frac{p/\rho}{u^2} + 1\right)_s \qquad (9\text{-}18)$$

The parameter to the left is known as a wall-force coefficient. From the terms to the right it is seen that it is a direct function of the ratios between the terminal flow-works and kinetic energies and also (for negligibly compressible fluids) that between the terminal areas (A_1/A), as $u/u_1 = A_1/A$.

It will be convenient to have relations paralleling eq. 9-17 but providing for situations which are less simple than that of fig. 9-12, and also ones expressing tangential and radial components of the wall-force. Channels and orientations representing such situations are indicated in figures 9-13 and 9-14, and are considered below. Note that in these the orientation of an approaching stream is taken to be the same as that of the channel walls at entry, as otherwise rather unaccountable stream disturbances will occur at this point; also that, although with some approximation, the orientation of the departing stream is likewise taken to be the same as that of the channels walls at exit.

(A) With relation to the channel of fig. 9-13, and for expressing first the wall-force components along the indicated x- and y-coordinates, by noting that $A \cos \lambda$ and $A \sin \lambda$ represent projections of areas A normal, respectively, to the x and y directions, the x- and y-coordinates,

$$F_{W,x} = (p_1 A_1 \cos\lambda_1 + \dot{m}\, u_1 \cos\alpha_1) - (pA \cos\lambda + \dot{m}\, u \cos\alpha) \qquad (9\text{-}19a)$$

Due care is evidently necessary with regards to the quadrant and corresponding algebraic sense of the functions of the several angles. It may frequently be that $\lambda = \alpha$, or that $\lambda = 90^\circ$ (and $\cos\lambda$ = zero).

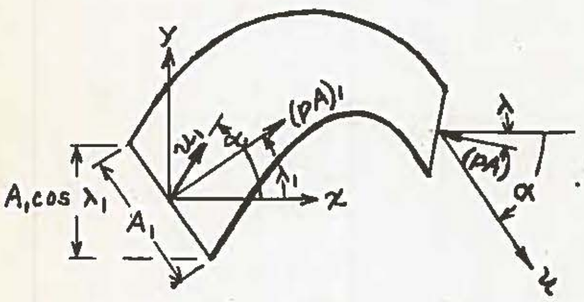


Fig. 9-13

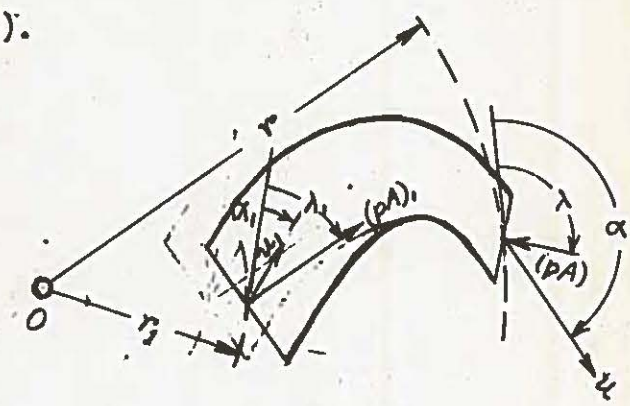


Fig. 9-14

By noting that $\dot{m} = \rho A\, u \cos(\lambda - \alpha)$ this relation may be put also in terms of a dimensionless <u>wall-force coefficient</u>, or

$$\frac{F_{W,x}}{\dot{m}\, u_1} = \left[\frac{p_1 \cos\lambda_1}{\rho u_1^2 \cos(\lambda_1 - \alpha_1)} + \cos\alpha_1\right] - \frac{u}{u_1}\left[\frac{p\cos\lambda}{\rho u^2 \cos(\lambda-\alpha)} + \cos\alpha\right] \qquad (9\text{-}19b)$$

The y-component of the wall force, and corresponding wall-force coefficient, are expressed in the same manner exsept that the sine function replaces the cosine in the terms $p_1 \cos\lambda_1$ and $p \cos\lambda$, and terms $\cos\alpha_1$ and $\cos\alpha$.

(B) With reference to fig. 9-14, and for expressing the <u>tangential and radial components of the wall force</u>, with angles λ and α now regarded as ones relative to tangents to arcs about O at relevant radii and points, and with the indicated orientations, a wall-force coefficient paralleling eq. 9-19b becomes

$$\frac{F_{W,\,tang}}{\dot{m}\, u_1} = \left[\frac{p_1 \cos\lambda_1}{\rho u_1^2 \cos(\lambda_1 - \alpha_1)} + \cos\alpha_1\right] - \frac{u}{u_1}\left[\frac{p\cos\lambda}{\rho u^2 \cos(\lambda-\alpha)} + \cos\alpha\right] \qquad (9\text{-}20)$$

The moment of the tangential component of the wall force ($M_{W,tang}$), about axial O, may well become of greater direct concern. In terms of a <u>moment coefficient</u>,

$$\frac{M_{W,tang}}{\dot{m}\, u\, r} = \frac{u_1 r_1}{u\, r}\left[\frac{p_1 \cos\lambda_1}{\rho u_1^2 \cos(\lambda_1 - \alpha_1)} + \cos\alpha_1\right] - \left[\frac{p\cos\lambda}{\rho u^2 \cos(\lambda-\alpha)} + \cos\alpha\right] \qquad (9\text{-}21)$$

A wall-force coefficient expressing the radial force component becomes

$$\frac{F_{W,\,rad}}{\dot{m}\, u} = \frac{u_1}{u}\left[\frac{p_1 \sin\lambda_1}{\rho u_1^2 \cos(\lambda_1 - \alpha_1)} + \sin\alpha_1\right] - \left[\frac{p\sin\lambda}{\rho u^2 \cos(\lambda-\alpha)} + \sin\alpha\right] \qquad (9\text{-}22)$$

9-11. Wall-force Relations for Flow of Gases. The preceeding relations are suitable for the flow of gases if the pressure change encountered is quite moderate, but become unsuitable if that is such as to produce a material change in the density of of the gas. But, as for a gas $p/\rho = RT$ and if the entry and departure temperatures of the stream may be anticipated by thermodyhamic analyses, for diatomic gases (such as air) at moderate pressure levels the wall-force coefficient α (for example) may be expressed by the relation

$$\frac{F_{W,x}}{\dot{m}\,u} = \frac{u_1}{u}\left[\frac{RT_1 \cos\lambda_1}{u_1^2\cos(\lambda_1-\alpha_1)} + \cos\alpha_1\right] - \left[\frac{RT\cos\lambda}{u^2\cos(\lambda-\alpha)} + \cos\alpha\right] \qquad (9\text{-}23a)$$

where $\dot{m}$ = mass-rate of flow, slugs per second;

R = gas constant, and 1716 ft.lbf/(slug x $^oF_{abs}$ or ^oR) for air;

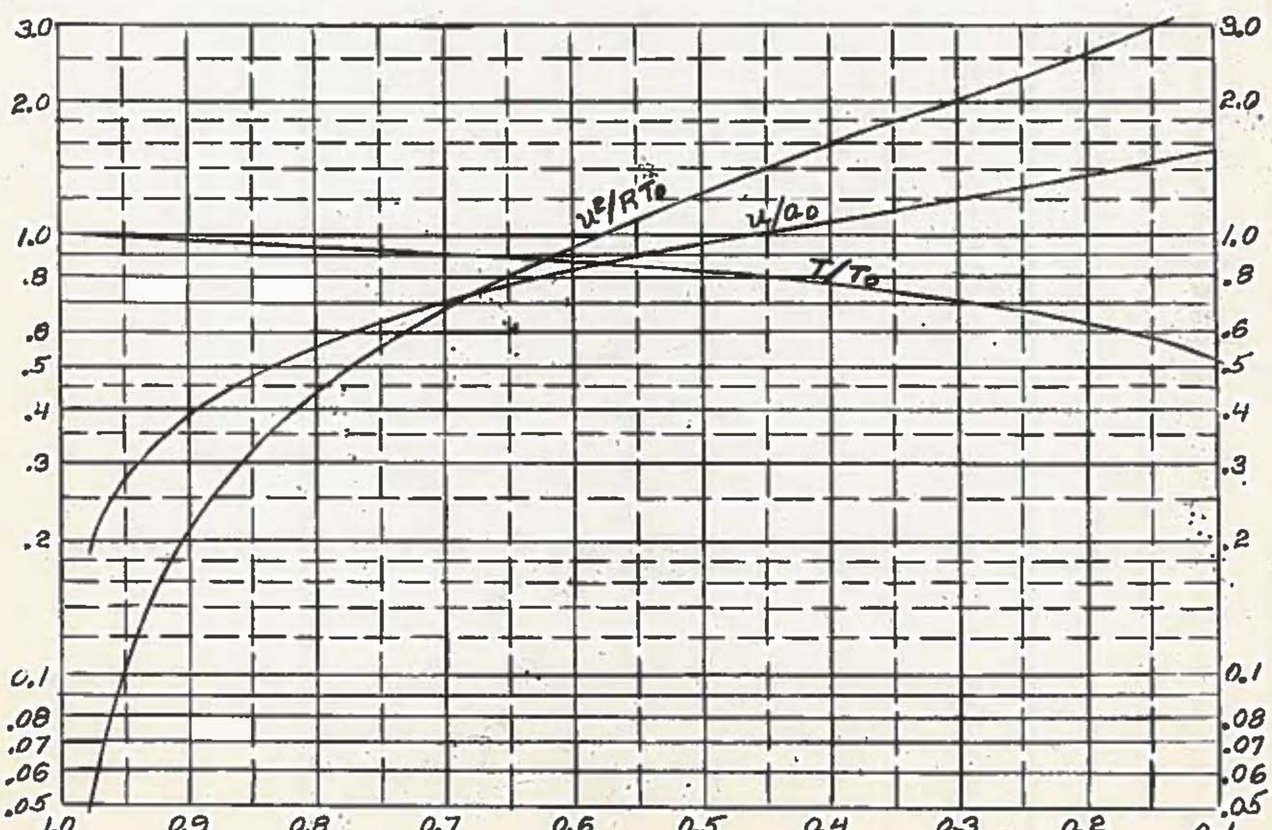
T_1 & T = absolute local temperatures in stream, ^oR;

u_1 & u = stream velocities, ft/sec.

As (local) pressures are much more readily measured than are the temperatures in a gas stream having considerable velocity, the following relation,

$$\frac{F_W}{\dot{m}\,a_o} = \frac{u_1}{a_o}\left(\frac{RT_o}{u_1^2}\frac{T_1}{T_o} + 1\right) - \frac{u}{a_o}\left(\frac{RT_o}{u^2}\frac{T}{T_o} + 1\right), \qquad (9\text{-}23b)$$

derived through thermodynamic analysis of idealized flow situations, is one in terms of items which are functions of relevant pressure ratios. For convenience the cosine (or sine) functions, adapting the relation to situations such as those of fig. 9-13 or 9-14, have been omitted. Again for diatomic gases at moderate



P/Po

Fig. 9-15

pressure and temperature levels, magnitudes of these items are provided in fig. 9-15 by its ordinate scale and the curves drawn to an abscissa scale of p/p_o, where p_o is an absolute <u>stagnation</u> pressure as perhaps measured by impact tube (Art. 7-7) in the entering stream. T_o is a like stagnation temperature, as measured reasonably closely by a thermometer the bulb of which obstructs the gas flow; also $a_o = 6.71\sqrt{RT_o}$ and is the velocity at which sound waves advance through the gas when at temperature T_o.

<u>Example 9-3</u>. Estimate the wall force ideally produced per unit mass-rate of through-flow in a straight nozzle to which air is entering at a stagnation temperature of 712°R, stagnation pressure of 102.7 psia and local pressure of 100 psia, if the discharge is to a region at 54 psia.

<u>Solution</u>. General items; $p_1/p_o = .975$, $p/p_o = .525$ and $a_o = 6.71\sqrt{1716 \times 712} =$ 1308 ft/sec. Entering the curves at these values of the pressure ratios, $u_1/a_o = .20$ and $u/a_o = .91$; $RT_o/u_1^2 = 1/.055 = 18.1$ and $RT_o/u = 1/1.17 = .855$; $T_1/T_o = .99$ and $T/T_o = .835$. Thus by eq. 9-23b; $F_{W,x}/\dot{m}_A^{a_o} = 1308\left[.20(18.1 \times .99 - 1) - .913(.855 \times .835 - 1)\right] = 2900$ lbf per slug/sec.

9-12. <u>Propulsion by Jet</u>. To note initially several considerations of interest historically and otherwise relating to the self-propelling of convenances:-

(a) With the invention of the steam or internal-combustion engine, and the mounting of these on a wheeled carriage supported by a firm rail or roadway, man became capable of directing some of the energy released by the combustion of fuels to the propulsion of a vehicle, through a turning of its wheels.

(b) A like ability does not exist when the conveyance is supported by a fluid and thus yielding medium, such as the marine vessel in or on water, or aircraft. There, for securing propulsive action, recourse is necessarily taken to a strategic application of the dynamic or wall force associated with the acceleration of a fluid stream, frequently a stream of the environmental fluid.

(c) Such adaptations of dynamic forces perhaps originated in the development, many centuries ago by the Chinese, of the early sky-rocket and is exemplified by their current use for the self-propulsion of missiles. However in the rocket the accelerated fluid is the hot gaseous product of the combustion of materials originally stored in body, and issueing in a <u>thermal jet</u>. Acceleration of an environmental fluid is accomplished by the <u>propeller</u>.

(d) The propeller is an outboard device which acts to give propulsive force by effecting the rearward acceleration of a relatively large stream of the water or air which surrounds the vehicle. Further attention is given to it in subsequent material. Although it is in effect a jet-propulsive device, that term is conventionally interpreted as signifying an arrangement in which the fluid stream is confined in a channel which is housed <u>within</u> the vehicle.

In the latter arrangements the wall-force of the preceding articles becomes a propulsive agency. The means employed for generating and maintaining the fluid stream are of various sorts and, for the more usual thermal jet, involve thermodynamic rather more then fluid-mechanic analyses.* However, if we may merely indicate a device by which the stream flow will be effected, the facilities of articles 9-8 and 9-9 may be employed for providing a sufficient introductory exposition of the principles and characteristics of propulsion by jet.

Referring to Fig. 9-15, representing a nacelle of an aircraft power installation, atmospheric air is there shown as entering an axially-oriented channel within it and as being brought, while in passage therein and by means of a compressor driven by a (turbine-type) internal combustion engine, to a pressure sufficiently exceeding atmospheric.

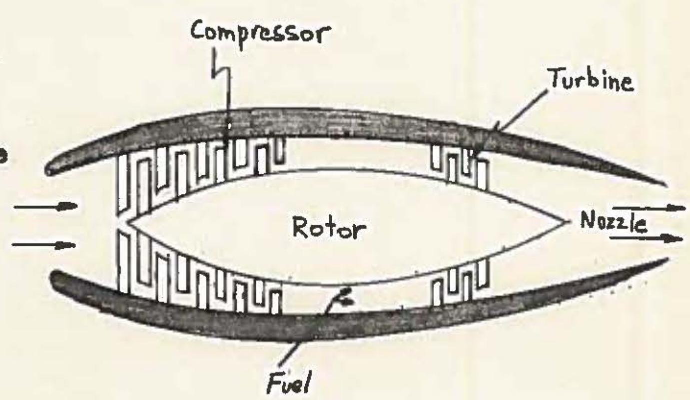


Fig. 9-15

The air, plus the products of the combustion of the fuel supplied to the engine, returns to the atmosphere through a convergent nozzle in which accelerative flow and expansion of the gases occurs. Recall (Art. 7-4) that a convergent nozzle enables expansion only to a pressure, temperature and velocity corresponding to attainment of unity

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* - Thermal jets are of several general characters, including (1) ones in which the oxygen required for the combustion of and energy release by a fuel is supplied by air inducted by a turbine-driven compressor and returned thence to the atmosphere, or the <u>turbo-jet</u>; and (2) <u>rockets</u>, in which the requisite energy release is by an exothermic reaction between materials which are initially stored within the device but are consumed and ejected as the reaction procedes.)

(1.0) Mach number. Following examples illustrate the use of foregoing facilities in connection with the analysis of such a jet-propulsion installation.

Example 9-2. The aircraft conveying the nacelle of Fig. 9-15 is operating at an elevation such that the atmospheric pressure and temperature are 8.0 psia and 0°F (460°R), and is advancing through the atmosphere at a velocity of 460 miles per hour, or 675 ft/sec. This will be taken as expressing also the relative velocity with which the air enters the channel.

Thermodynamic analyses will indicate that, at the exit section of the nozzle and under typical operating conditions, the air (plus combustion products) would depart at about 12.5 psia and 1375°R, and a rearward relative velocity of 1780 ft/sec.

Compute the component of the resultant wall-force acting on the interior surfaces of the channel, and contributing to the forward motion of the nacelle and its carrier, per unit mass of air passing through it per second; also the entry and exit areas of channel for combatibility in the above specifications.

Solution. Neglecting for simplicity the minor relative mass of the fuel combustion-products departing with the air stream, by Eq. 9-23,

$$\frac{F_{w,x}}{\dot{m}} = 675\left(\frac{1716 \times 460}{(675)^2} \times 1.0 + 1.0\right)$$

$$- 1780\left(\frac{1716 \times 1375}{(1780)^2} \times 1.0 + 1.0\right)$$

$$= 1840 - 3100 = -1260 \text{ lbf/(slug per sec)}$$

$$\text{or } = -39.2 \ \frac{\text{lbf}}{\text{lbm/sec}}$$

$$A_{1,x}/\dot{m} = \frac{1}{u_{1,x}\,\rho_1}, \quad \text{or} \quad \frac{RT_1}{p_1 u_{1,x}}$$

$$= (1716 \times 460)/(8 \times 144 \times 675) = 1.015 \ \text{ft}^2/(\text{slug per second})$$

$$\text{or } = .031 \ \text{ft}^2/(\text{lbm per sec})$$

$$A_x/\dot{m} = (1716 \times 1375)/(12.5 \times 144 \times 1780) = .737 \ \frac{\text{ft}^2}{\text{slug/sec}}$$

$$\text{or } = 22.7 \ \frac{\text{ft}^2}{\text{lbm/sec}}$$

The negativeness of F_w denotes as usual that it is oriented in a direction opposite to that of the stream or jet. This force necessarily equals in magnitude, but opposes, the aggregate of the forces resisting the forward motion of the nacelle and the craft carrying it. These include:

(a) the axial component of its weight (F_g), if ascending;

(b) its inertial resistance to any acceleration (F_i);

(c) the fluid-friction force (F_f), acting through the boundary layer and resisting its forward motion;

(d) a drag-force attributable to pressures exceeding atmospheric on those exterior surfaces with forwardly-facing area components, resulting from the forces necessary to displace the environmental fluid as the craft advances, and to typically less-than-atmospheric pressures on rearward-facing ones.

That portion of this drag due to nacelle alone may be represented as $\int (p - p_{atm})\, dA_x + p_{atm}(A_x - A_{1,x})$ where items $A_{1,x}$ and A_x are the axial projections of the apertures forming the channel entry and exit sections;* and as $\sum \int (p - p_{atm})\, dA_x + p_{atm}(A_x - A_{1,x})$ for the entire structure.

The equilibrium equation associating all forces is thus

$$\left[F_{w,x} + p_{atm}(A_x - A_{1,x})\right] + F_p + F_i + F_f + \sum \int (p - p_{atm})\, dA_x = 0$$

Here the bracketted term is customarily and suitably regarded as a resultant <u>propulsive force</u> on the flow channel <u>plus</u> nacelle, F_p. In that term items of the form $p_{atm}A$ may be put in the form $p_{atm}(\dot{m}/u\rho)$ or, for gases, $\dot{m}(RT/u)(p_{atm}/p)$. The propulsive force may thus be expressed as

$$F_p = F_{w,x} - \dot{m}\left(\frac{RT_1}{u_1}\right)\left(\frac{p_{atm}}{p_1}\right) + \dot{m}\left(\frac{RT}{u}\right)\left(\frac{p_{atm}}{p}\right)$$

A corresponding propulsive-force coefficient becomes, at λ_1, λ, α and $\beta = 0$

$$\frac{F_p}{\dot{m}u_{1,x}} = \left[\frac{RT_1}{u_1^2}\left(1 - \frac{p_{atm}}{p_1}\right) + 1\right] - \frac{u}{u_1}\left[\frac{RT}{u^2}\left(1 - \frac{p_{atm}}{p}\right) + 1\right] \qquad (9\text{-}24)$$

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* - Such representation may be justified by reference to Fig. 9-16. There, for a nacelle arrangement such as that of Fig. 9-15, a reasonably representative distribution of the total, or absolute, pressures (p) on the <u>exterior</u> surfaces is shown by plotting them against the axial

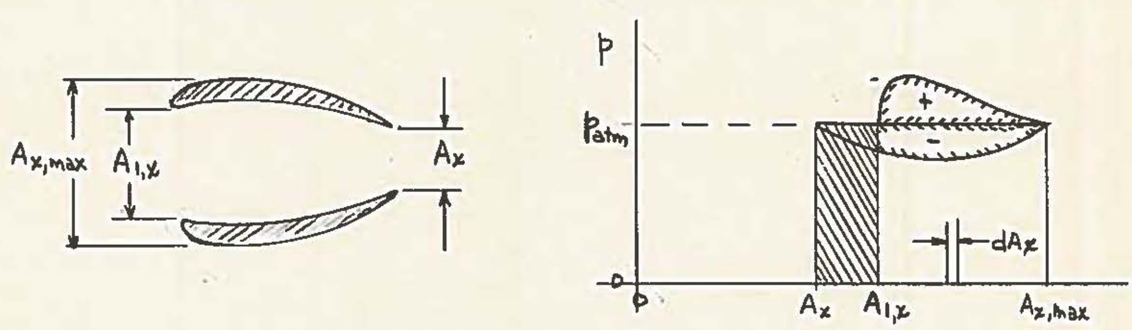


Figure 9-16

projections of those surfaces. Of the total pressures the atmospheric portion (p_{atm}) operates on all surfaces, and is assistant on rearward-facing ones but restraintive on forward-facing ones. The product $p_{atm}(A_x - A_{1,x})$ so represents, in this instance, a net assistant influence of p_{atm}. The pressure excesses (+), relative to atmospheric, on the forward-facing exterior surfaces, and the deficits (−) on the rearward-facing ones, act cumulatively in producing the drag-force in question, and is representible as $\int (p - p_{atm})\, dA_x$.

In the above evaluations of the wall force, acting on interior surfaces of the flow channel, the velocities involved were necessarily ones relative to that channel. For subsequent purposes of energy accountings, whereby the efficiency with which the propulsive force may actually be utilized, attention is necessarily given to velocities relative instead to the environmental fluid through which the channel and its housing is advancing. These will be distinguished by symbol U. To illustrate, for the jet stream leaving the channel

$$U_j = u - U$$

where U = the velocity of advance of channel-and-craft in the environment;

u = the rearward and relative velocity of departure of equation 9-24; and

U_j = the residual rearward velocity of the jet stream at point of departure, but relative to the environmental fluid.

For related energy accountings it is convenient to consider also energy-rates, or power; as expressed in terms of force-times-velocity or of mass-rate of flow times (kinetic) energy per unit mass, both having the basic dimension ML^2T^{-3}. The power usefully employed in propelling the engine and craft correspondingly becomes

$$\text{Propulsive power} = F_pU$$

Recall that F_p represents both the above actuating force and the aggregate restraining force (due to the above F_g, F_i et cetera) resisting the advance of nacelle-and-carrier. In steady level flight this is approximately proportion to U^2.

A convenient dimensionless parameter associating F_p, the mass-rate ($\dot{m}$) of through-flow and departure of the fluid, and relevant velocities is a

$$\text{Propulsive power coefficient} = (F_pU)/(\dot{m}U_j^2)$$

As the kinetic energy of the leaving jet is promptly dissipated in shock-wise disturbances and/or turmoil in the atmosphere, a measure of the energy and power so "lost" is

$$\text{Dissipated power} \simeq \dot{m}\, U_j^2/2$$

Reasonably regarding the sum of these powers as an ideally utilizeable aggregate, the quotient of the propulsive power divided by this sum is similarly regarded as an index of the mechanical efficiency of the actual propulsion, or

$$\text{Mechanical efficiency of propulsion, } \eta_{prop} = \frac{F_pU}{F_pU + \dot{m}U_j^2/2} \qquad (9\text{-}25)$$

The following example illustrates the utilization of these relations in connection with the conditions and results of example 9-2 .

Example 9-3 . For the conditions and utilizing the results of example 9-2 compute :

(a) the mass-rate and volume-rate of air intake required for securing a thrust (i.e., propulsive force) of 2000 lbf;

(b) the associated propulsive power and propulsive power coefficient;

(c) the rate of kinetic energy dissipation in departing jet stream;

(d) the mechanical efficiency of propulsion.

Solution.

(a) $F_p/\dot{m} = 675\left[\frac{1716 \times 460}{675^2}(1.0-1.0) + 1.0\right]$

$$- 1780\left[\frac{1716 \times 1375}{(1780)^2}\left(1.0 - \frac{8}{12.5}\right) + 1.0\right] = -1605 \frac{\text{lbf}}{\text{slug/sec}}$$

$\dot{m} = \frac{2000}{1605} = 1.246$ slugs/sec $= 40.1$ lbm/sec

$\dot{V} = 1.246 \times (1716 \times 460)/(8 \times 144) = 850$ ft^3/sec

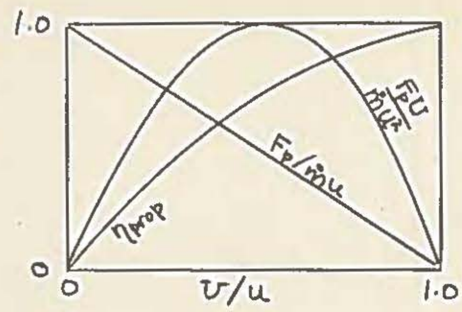
(b) Propulsive power = 2000x675 = 1.35×10^6 ft.lbf/sec (= 2,543 hp)

Power coefficient = $\frac{1.35 \times 10^6}{1.246(1780-675)^2} = 0.888$

(c) Dissipated power = $1.246(1780-675)^2/2 = .761 \times 10^6$ ft. lbf/sec

(d) Mechanical eff. of propulsion = $1.35/(1.35 + .761) = 0.64$

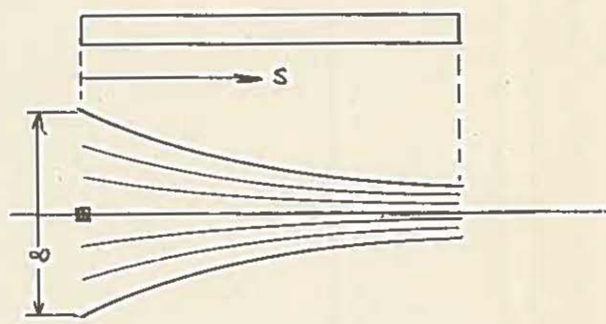
Analyses serving to indicate for thermal-jet propulsion the consequences of departures from the above conditions would involve thermodynamic considerations beyond the scope of this material. However a significant observation in connection with Eq. 9-25 is that, although 100% propulsive efficiency would appear to be securable if $\dot{m}U_j^2$ were zero, this corresponds to the quite futile situation of zero magnitudes of $\dot{m}$ and/or U_j and a consequent zero propulsive effect. At the other extreme, although a maximum propulsive force is obtained if the unit is not permitted to advance (as in frequent test procedures), the propulsive power is then zero. It may be shown, but at this point will not be verified, that relationships between various of the above variables are somewhat as indicated in Fig. 9-17.



9-12 Problems.

1. Write an expression for the transverse energy distribution in laminar flow through a circular pipe of radius R, Fig. 9-1(b). Calculate the radial location of the point of maximum energy gradient. Ans. $R/\sqrt{3}$

2. A two dimensional horizontal nozzle as shown is to accelerate frictionless water from zero velocity under the influence of a constant axial pressure gradient of $10\ \frac{\text{lbf}}{\text{in}^3}$. For a particle on the center line, initially cubical in shape,



a) Find expressions for the velocity of the particle as a function of distance S, and for the velocity and kinetic energy gradients.

b) Find the shape of the particle when 5" downstream.

3. A number of radially adjacent streams of water flow in concentric circular horizontal paths, forming a common stream of 1 foot inner and 3 ft. outer radius.

a) Determine the difference of pressure and of velocity between the two radial limits, and also of the aggregate mechanical energy per unit mass ($\frac{p}{\rho} + \frac{u^2}{2}$), if at the 1 foot radius the linear velocity is 10 ft/sec and the velocity varies so that $u_t r =$ constant.

b) Determine the same items if in some manner the angular velocity throughout the stream is caused to be the same as that at the 1 foot radius.

4. An observed "super" vortex has a velocity distribution of the form $\omega = C r^{1/2}$ or $u_t = C r^{3/2}$. Calculate the radial pressure distribution.

5. A wind velocity of 100 miles per hour is observed at a point in a hurricane estimated as being 50 miles from the axis or "eye" of the storm. Calculate the maximum velocity in the storm if it is assumed to occur at a dis-

tance of 10 miles from the center, and that at a point 5 miles from the center.

6. Find the barometric pressures at radii of 5, 10 and 50 miles from the center of the storm of Prob. 9-5 if an observation at 200 miles indicates a pressure of 28.8" of mercury, and a temperature of 90°F which may be assumed constant over the entire hurricane.

7. Water flows through a tee in a horizontal pipe system. The velocity in the stem of the tee is 15 ft/sec and the diameter is 12". The branches are each 6" in diameter. If the pressure in the stem is 20 psig. Calculate the magnitude and direction of the wall force on the tee.

8. A 45° reducing elbow carried fresh water at the rate of 120 $\frac{ft^3}{min}$. The pressure at the inlet end which is 6" in diameter is 20 psig. The exit end is 3" in diameter, and Ø is 150 $\frac{ft\ lbf}{slug}$. Taking the inlet direction as a reference, calculate the components of the wall force in the 0 and $\pi/2$ directions and their resultant.

9. A so called flying platform keeps aloft by discharging air vertically downward at an average velocity of 600 ft/sec. In order to arrive at an order-of-magnitude estimate of the required mass flow of air for a load of 50 lbs per sq ft of discharge area, assume that the air is incompressible and is discharged at a temperature of 100°F and a pressure of .1 psig. Estimate the required flow per ft^2 of discharge area to enable the device to ascend at the rate of 600 ft/min.

10-1. Foreword. Engineering devices for extracting energy mechanically from a liquid or for imparting such energy to it, or pumping, fall into two general classes. Positive-displacement machines, as the name implies, operate simply by direct physical displacement of the fluid through the agency of a solid piston caused / to reciprocate along an enclosing and suitably valved cylinder, or of equivalent rotating elements such as meshing gears or lobes, advancing screws et cetera. These are utilized more frequently for pumping, and are particularly well adapted to conditions where pressure differences or volumetric flow rates require close control, or if the viscosity of the liquid is rather high. But the type is characterized by relatively moderate flow capacities for a given size of machine. Typical applications are fuel oil pumps, hydraulic servo-mechanisms, hydraulic hoists or jacks, variable-speed transmissions etc. As these machines do not utilize essentially hydrodynamic phenomena, they will not be given further attention.

More recently, and in situations requiring relatively compact equipment capable of handling larger rates of flow, hydrodynamic equipment utilizing dynamic forces such as are associated with the forced spiral or helical vortex (Art. 9-9) have come almost exclusively into use. Pumps of this character are in general classifiable as centrifugal and propeller types, or as of mixed-flow type if a combination. Turbines, more commonly employed for procuring power from the energy of water which has been supplied as rain or snow fall and accumulated in elevated reservoirs, are of impulse, reaction, propeller and mixed-flow types. Ingenious pump-and-turbine combinations form the hydraulic couplings and torque-converters which have become rather standard equipment in the automobile and find many industrial applications.

PUMPS

10-2. Centrifugal Pumps. The centrifugal type of pump consists essentially of a suitably formed casing which houses a multi-celled rotor, or impeller, and serves also to conduct the fluid stream to and thereafter from the rotor. The latter is driven by a shaft which extends through the casing to an exterior motor. By providing in the delivery portion of the casing a progressively increasing stream area, much of the kinetic energy of the streams leaving the rotor cells is therein converted to "pressure head" (i.e., flow work). Fig. 10-1 indicates by cut-away

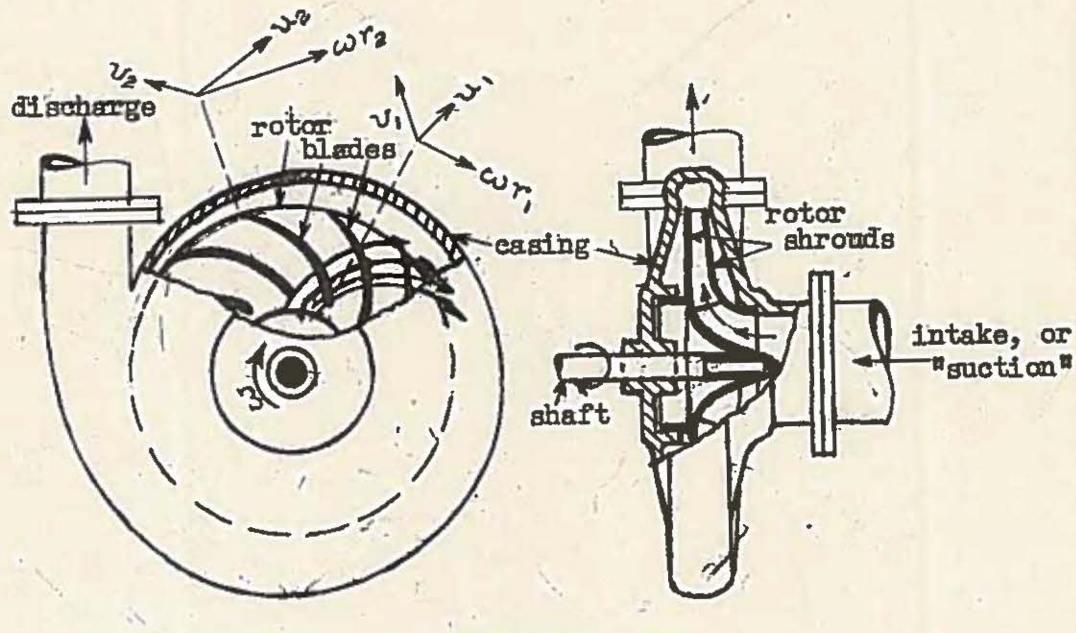


Fig. 10-1 Centrifugal Pump

views the general arrangement of casing, impeller and shaft for a pump having a single rotor and inflow to only one side of that. For enabling a greater pressure rise in a single, but multi-stage, machine a number of like rotors may be mounted on a single shaft and in a single composite casing. That portion of the casing which surrounds each rotor is individually so formed as to provide suitably for the flow to and from that rotor, but channels are also provided through which the fluid may pass progressively from one rotor to the next.

In the figure the general channel formed by the shrouds bounding the rotor is seen to be divided into a number of cells by blades which extend between the shrouds and which serve to enforce on the streams while enroute through the cells the composite of a tangential motion about the rotor axis and an outward flow. As there represented the blades have a rearward curvature relative to the direction of rotation of the impeller, which is in contrast to the radial blading of fig. 9-9, but in general the flow pattern through each cell is still of forced-spiral vortex character. However quite frequently the blades are further so twisted as to terminate at inner end with a practically radial orientation, in which case the flow pattern progresses from one of forced-helical to one of forced-spiral vortex character.

Fig. 10-2 aims to represent a rotor with this character of blades. Blading has also been employed having a radial or even forwardly-curved contour near its exit end, but with some disadvantage in performance characteristics.

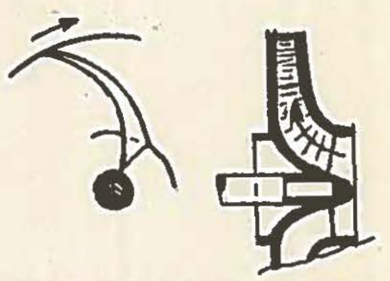


Fig. 10-2 Mixed-flow Rotor

In agreement with the characteristics of the forced spiral vortex, as developed in art. 9-9(b), the aggregate mechanical energy attributable per unit mass to any passing particle of the stream through the rotor (turning at angular velocity ω) may be regarded as a direct function of r^2. Also, in the individual cells and along a given arc about the axis, the velocities (and kinetic energy) progress from a minimum at the face of an advancing blade to a maximum at the back of its retreating neighbor. The consequent superior pressure at the face of a blade, relative to that at its back, originates the wall-forces causing the torque necessary to actuate the impeller and is the agency whereby the desired energy input is furnished to the stream while en route through the rotor cells.

Although the above describes the general situation, precise kinematic analysis is again infeasible. Also the influences of viscosity and turbulence and the invariable trend towards the resumption of irrotational flow produce uncertain departures from forced-vortex configuration. For such reasons and with some practical advantage, but admitted unreality, an endeavor is customarily made to formulate the torque and power for actuating the impeller on the premise that the velocities of the fluid particles as they leave a rotor cell may be represented approximately by a single vector of mean velocity $\bar{u}$, and similarly as they enter.

For such formulations adaptations of equations 9-20 and 21 may be developed by observing that -

(a) In contrast to the channel of fig. 9-14, to which those equations relate, for the impeller the angles λ are evidently 90° and their cosine thus zero, so that in the modified equations items involving terminal pressures (p) will disappear (i.e., the "pressure forces" are tangentially inoperative).

(b) As the cells of the impeller are in motion about the shaft (in contrast to the stationary channel of fig. 9-14), and the terms in cos α of the primary relations express the tangential components of the absolute velocities of the stream arriving at or leaving the channel, to express these it is necessary to account for the influences of the rate of turning and size of the rotor, of the orientation of the blade surfaces, of the volumetric rate and transverse areas of the stream, and of their consequent velocities relative to the rotor.

For present purposes it is sufficient to give attention only to rotors and blading of such configuration that at entry to the cells the streams have presumably a negligible axial component of absolute velocity. For such, and referring to the vector diagrams of Fig. 10-1 or 3, the terms of eq. 9-20 involving the tangential component of the absolute leaving or entering velocities, $\bar{u} \cos\alpha$, may be expressed as:

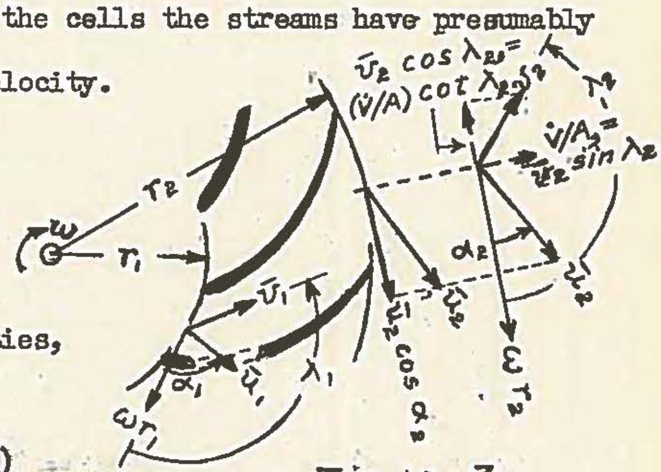


Fig. 10-3

$$\bar{u}\cos\alpha = \omega r + (\dot{V}/A)\cot\lambda \qquad (10\text{-}1)$$

where ω = turning speed of rotor, radians per second;
r = radial distance to relevant end of blade;
$\dot{V}$ = volume-rate of flow through rotor ($= \dot{m}/\rho$);
A = net area of cylindrical annulus traced by outer or inner ends of blades, with $\dot{V}/A$ thus expressing the radial component of of velocity of the stream leaving or entering the rotor cells;
λ = angular orientation of the blade surfaces at exit or entry points, relative to the tangential direction of their advance.

It will appear from fig 10-3 that $(\dot{V}/A)\cot\lambda$ expresses nominally the tangential component of the relative velocities with which the streams leave or enter the rotor cells.* But in actuality their relative direction may differ appreciably from the "blade angle" (λ), due to influences of the geometry of the flow fields through which they have arrived or to which they depart.

Adapting eq. 9-21 to present purposes in conformity with the foregoing but expressing the torque (T) required to drive the rotor, rather than the opposing moment generated by the through-flow of the streams,

$$\begin{aligned} T &= -F_{w,tang}\,\bar{r}, \\ &= \dot{m}\left[r_2(\bar{u}_2\cos\alpha_2) - r_1(\bar{u}_1\cos\alpha_1)\right] \\ &= \dot{m}\left[\left(\omega r_2^2 + \frac{\dot{V}r_2}{A_2}\cot\lambda_2\right) - \left(\omega r_1^2 + \frac{\dot{V}r_1}{A_1}\cot\lambda_1\right)\right] \qquad (10\text{-}2) \end{aligned}$$

Or arranged in terms of dimensionless parameters,

$$\frac{T}{\dot{m}\omega r_2^2} = 1 - \frac{r_1^2}{r_2^2} + \frac{\dot{V}}{\omega r_2 A_2}\left(\cot\lambda_2 - \frac{A_2}{A_1}\frac{r_1}{r_2}\cot\lambda_1\right) \;** \qquad (10\text{-}2a)$$

(* – The general character of the absolute path of the stream passing through a moving cell, and of the velocity distribution within it, is somewhat as represented by the family of stream-lines appearing in Fig. 10-1.)
(** – Products having the form $\dot{m}\omega r^2$ are frequently described as expressing the "rate of angular momentum".)

The power input correspondingly required equals ωT and may be of more direct concern, or

$$\frac{P_{dyn}/\dot{m}}{\omega^2 r_2^2} = 1 - \frac{r_1^2}{r_2^2} + \frac{\dot{V}}{\omega A_2 r_2}\left(\cot\lambda_2 - \frac{A_2}{A_1}\frac{r_1}{r_2}\cot\lambda_1\right) \qquad (10-3)$$

Note that $P_{dyn}/\dot{m}$ nominally expresses only the energy input to the impeller per unit mass of fluid delivered, or ${}_1w_2$ of the energy equation.* As a whole the relation serves at least to suggest the mutual influences on the requisite power as exerted by the rotary speed, size, proportions and blading orientation of the impeller, and the rate of delivery of the fluid. However the over-simplifications accepted in developing it make it only broadly indicative of such influences. The design of impellers and anticipation of their performance is in fact rather less a science based on such analyses than it is an art evolved through the organization of experience.

Considerations of more direct concern to pump supplier or user, and more readily determinable ones, include -

(a) The total power in fact required for actuating the shaft, P_{shaft}, exceeding the rate of energy transfer to the fluid through the agency of the impeller by the amount dissipated by friction between moving and stationary parts of the pump.

(b) The useful power output of the pump as expressed by the product of the measured or desired mass-rate of delivery times the excess of the aggregate, per unit mass, of the flow-work, kinetic and geopotential energies at the discharge-pipe connection over that at intake connection. This power is conventionally known as the hydraulic power output, P_{hyd}, and is expressed as

$$P_{hyd} = \dot{m}\left[(p_d - p_i)/\rho + (u_d^2 - u_i^2)/2 + g(z_d - z_i)\right] \qquad (10-4)$$

where subscripts d and i denote discharge and intake points. The bracketted sum is described as the total head delivered. This useful power is less than that furnished the impeller by the rate of energy dissipation by fluid friction and turbulence (${}_1\phi_2$) encountered in the passage of the fluid through the pump.

(c) The ratio P_{hyd}/P_{shaft}, known as the overall efficiency (η_{pump}) of the unit.

(d) The relation actually existing between the rate of delivery of which the pump is capable when operating at a given speed, and the range of pressure or of total head through which it is concurrently required to deliver.

The curves of Fig. 10-4 are representative of those by which the inter-relation between the above items is commonly portrayed, for a specific pump when operating at some constant and stated rotative speed. Families of like curves may, however, be shown for operations at several speeds. A more generalized graphical method of performance correlation, employing dimensionless parameters, will be considered in

(*- But it provides no basis for estimate of the energy-dissipation term (${}_1\phi_2$) of that equation, or thus of the energy usefully imparted to the fluid and enabling, primarily, its delivery against a superior discharge pressure.)

Art 10-4. The units of the several scales of the figure are not consistent ones, but are ones customarily employed by the hydraulic engineer. All of the curves relate to the single abscissa scale but individually only to the suitable ordinate scale.

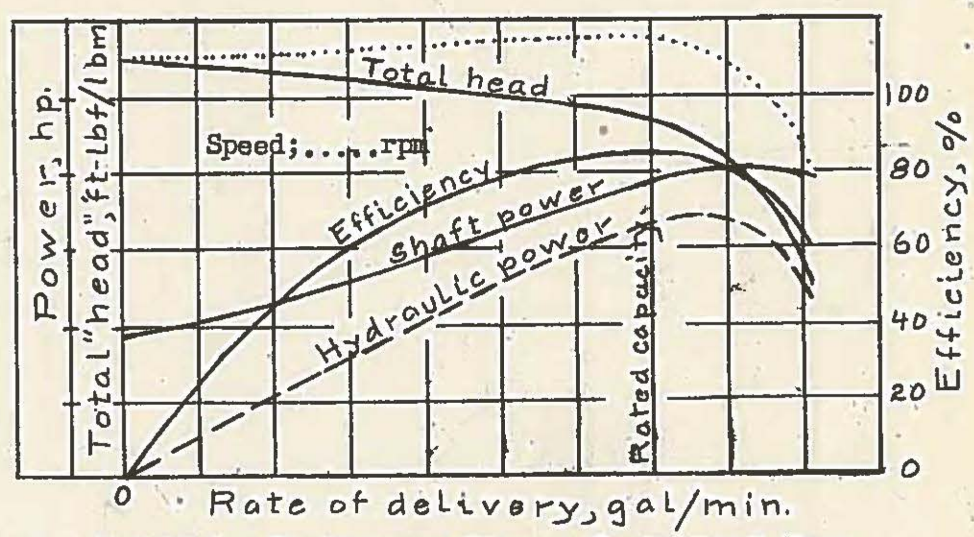


Fig.10-4. Performance Curves, Centrifugal Pump

The solid head-capacity curve of the figure is typical for blading which is backwardly-curved throughout, as in Fig. 10-1 or 3; a temporarily "rising" head such as is indicated by the dotted curve is obtainable with blading which is forwardly curved near the outer end, but a rotor with such blading may be unstable in operation due to the evident ability to deliver against the same head at two rates. The delivery capacity for which a pump is "rated" when operating at a given speed is normally taken as that at which maximum efficiency is obtained.

The vertical separation between the curve of shaft-power input and the (broken-line) curve of hydraulic power output is a measure of the rate of energy dissipation in and by the pump, due jointly to the reasonably constant mechanical friction between moving parts and to fluid friction and turbulence effects. The last increases progressively with departure from delivery rates corresponding to rated capacity, and is attributable to the inability of blading with fixed orientation to provide favorably for inflow of fluid to the impeller cells at such rates. The curves of the figure are for operating conditions in which <u>cavitation</u> (Art. 10-5) is not encountered.

10-3. Propeller-type Pump. This type of pump is so known because of features of form, purpose and principle which are similar to those of the marine propeller, such as that of the familiar outboard motor or of the large commercial vessel. Both operate through the action of dynamic forces in generating a forced-helical type of vortex flow in the stream passing theough the impeller. However the pump impeller rotates in a stationary casing which closely surrounds it and through which the fluid is causedto advance axially, while the marine propeller operates in an unconfined liquid field in which it itselfadvances. This type of pump is best adapted to services in which relatively larger delivery rates are required but against only quite moder- head, and to situations in which the impeller may be located below the sur- face of the body of liquid from which suction is taken.

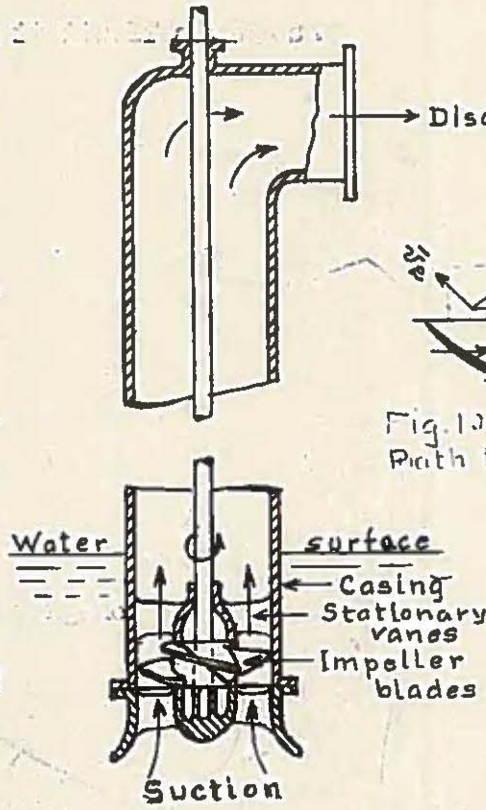


Fig. 10-5. Propeller-type Pump.

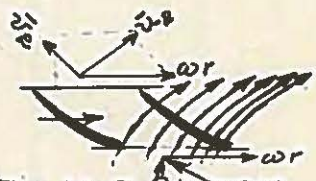


Fig. 10-6. Stream Path thru Cell
Fig. 10-6

Fig. 10-5 indicates the general arrangement of the pump, with rotating shaft and an impeller consisting of a number of suitably contoured foils projecting radially from a central hub. Surrounding these is a cylindrical casing through which the liquid approaches and leaves the impeller, with the provision also of stationary radial vanes which serve to guidethe stream advantageously to and from the cells formed by the impeller blades.

The kinematics and dynamics of the streams passing through the cells are too complex to justify present attention. However Fig. 10-6 may indicate sufficiently the general nature of the absolute path of and velocity distribution in the stream passing though an individual cell as it moves about the shaft axis. Primary attributes of the stream as so portrayed are (a) the superior pressure associated with the lesser velocities at the advancing surface of a foil, and (b) the con- verse greater velocity and lesser pressure at the retreating surface of its neighbor. The tangential component of the net force resulting from the difference in pressures between those at the two surfaces of a foil is that which necessitates the supplying of torque for turning the impeller and provides for the furnishing of energy to the stream. The axial component is the agency enabling delivery of the stream against a superior discharge pressure.

Typical curves associating the total head and efficiency and the rate of delivery for this type of pump appear in the figure of the following article, with comparable ones for the centrifugal pump.

10-4. Generalized Performance Correlation, Hydrodynamic Pumps. Through the technique of dimensional analysis, and the concept of dynamic similarity of flow through geometrically similar channels (Arts. 6-5 etc.), the performance of all of a family of geometrically similar pumps may be correlated to considerable advantage, and (in principle) from data obtained from test of a single representative or model. While such data provide no information concerning the effects of change in the geometric proportions or orientations, the practical benefits are great when information which is so correlated is available for various types of pumps and one wishes to select a type and size suitable for a given service, and to anticipate its performance under various operating conditions.

It will be recalled (Art. 6-6) that a requirement for such correlation is first to list all variables which by experience or intuition may be expected to influence the performance, and the basic dimensions of those variables. Here again the dimension of mass will be more convenient than that of force. Variables which will be useful measures of the subject performance need also be recognized.

For an operating pump relevant variables, and their basic dimensions, are as follow.

For the liquid

density of liquid, ρ; ML^{-3}
viscosity, μ; $ML^{-1}T^{-1}$

For the pump

rotor diameter, d; L
rotative speed, ω; T^{-1}
proportions and orientations of parts; $M^0L^0T^0$

Performance items

Rate of delivery, $\dot{V}$ (or $\dot{m}/\rho$); L^3T^{-1}
Energies per unit mass of liquid delivered (or "head"), Δe; L^2T^{-2}

The net or hydraulic power output (P_{hyd}; ML^2T^{-3}) is an equally significant performance item, but is recalled to be the product of the mass-rate of delivery and the useful energy per unit mass. The rate of energy impartment to the fluid by the impeller (P_{rotor}) and the gross power introduced via the shaft (P_{shaft}) are further important items, exceeding successively the power output by the rates of energy dissipation in fluid friction and turbulence and in mechanical friction. The correlation of these is considered subsequently.

As six dimensional variables appear above, and three basic dimensions are involved, the rules of the π-method of procedure (Art. 6-6b) require at least three dimensionless combinations of these, or π's, with four variables initially included when involving any of the three. So proceeding -

(A) Associating ρ, μ, d and ω,

$$\pi_1 = \rho\,\mu^a d^b \omega^c = (ML^{-3})(ML^{-1}T^{-1})^a(L)^b(T^{-1})^c, \text{ and } = M^0L^0T^0.$$

Thus, for dimensional homogeneity

in M, $1 + a = 0$ and $a = -1$;
in T, $-a - c = 1 - c = 0$ and $c = +1$;
in L, $-3 - a + b = -3 + 1 + b = 0$ and $b = +2$.

Or $$\pi_1 = \frac{\rho d^2 \omega}{\mu}, \text{ or } \frac{\rho d (d\omega)}{\mu} \qquad (10\text{-}4)$$

But note that in this parameter the product $d\omega$ has the significance of a (tangential) velocity, as perhaps at the periphery of the rotor, and π_1 is thus in effect an equivalent of the Reynolds index ($du\rho/\mu$, or N_R). Also recall that for the clearly turbulent flow associated with high magnitudes of this index a considerable variation of it has been seen to have minor influence on the dynamic character of the flow (as was indicated in fig. 6-3). Thus, as has been experimentally verified and except as very considerable change is introduced through operation of a pump at very low speeds or with liquids of greatly differing viscosity or density, the influence of its variation may normally be neglected in pump-performance correlations.

(B) To develop a second parameter by associating $\dot{m}$, ρ, d and ω,

$$\pi_2 = \dot{m}\,\rho^a d^b \omega^c = (MT^{-1})(ML^{-3})^a(L)^b(T^{-1})^c, \text{ and } = M^0L^0T^0.$$

Here for homogeneity in M, $1 + a = 0$ and $a = -1$;
in T, $-1 - c = 0$ and $c = -1$;
in L, $-3a + b = 3 + b = 0$ and $b = -3$.

Thus $$\pi_2 = \frac{\dot{m}/\rho}{d^3\omega}, \text{ or } \frac{\dot{V}}{d^3\omega} \qquad (10\text{-}5)$$

When, in evaluating this parameter, the volume-rate of delivery is expressed (and per side of an impeller having intake to its two sides) in the common industrial unit of gallons per minute $(\text{gpm})_A$, the diameter in feet and the rotative speed in revolutions per minute (rpm), it is referred to as a <u>capacity coefficient</u> or <u>specific capacity</u>.

(C) Introducing the remaining variable Δe, or the total "head" delivered,

$$\pi_3 = \Delta e\,\rho^a d^b \omega^c = (L^2T^{-2})(ML^{-3})^a(L)^b(T^{-1})^c, \text{ and } = M^0L^0T^0.$$

Here for homogeneity in M, $a = 0$;
in T, $-2 - c = 0$ and $c = -2$;
in L, $2 - 3a + b = 2 - 0 + b = 0$ and $b = -2$,

and $$\pi_3 = \Delta e/d^2\omega^2 \qquad (10\text{-}6)$$

To summarize, the interpretation of the foregoing is that

$f(\pi_2,\pi_3)$ = constant for all of a family of geometrically similar pumps.

That is, if, instead of a head-capacity curve such as that of fig. 10-4 for a particular pump operating at a single speed, corresponding test values of π_3 are plotted against simultaneous ones of π_2, the resulting and like-appearing curve may be regarded as closely acceptable for all geometrically similar pumps operating at reasonable speeds.

These parameters do not contain the equally important item of the energy dissipation per unit mass due to fluid friction and turbulence (ϕ), but note that one paralleling π_3 and of the form $\phi/d^2\omega^2$ would be similarly dimensionless and similarly acceptable as a function of π_2. Furthermore, however, as the hydraulic efficiency (η_{hyd}) may be written as $\Delta e/(\Delta e + \phi)$ and here both the numerator and the denominator are thus functions of π_2, their ratio is equally so. Relative finally to the energy dissipation through mechanical friction, and the associated overall efficiency, that loss may not in principle be correlated with the above other items through dynamic similarity considerations. But in a well constructed and maintained pump it is sufficiently minor in influence that, with some approximation but considerable convenience, the overall efficiency is customarily taken in practice as also sufficiently closely a function of π_2.

The primary coordinates of fig. 10-7 are correspondingly that of specific capacity as abscissa and those of specific head (per stage) and overall efficiency as ordinates. To these coordinates there are also drawn representative specific head and overall efficiency curves for typical centrifugal and propeller types of pumps. These curves serve further to indicate the rather distinctive natures and ranges of their performance characteristics.

As in similar prior circumstances, it is convenient to have available in fig. 10-7 (skewed) ordinates and scales of several supplementary and dimensionless parameters which are joint functions of the primary coordinates (π_2 and π_3) and are thus single-valued for given simultaneous magnitudes of those. The composition and form of such are as follow.

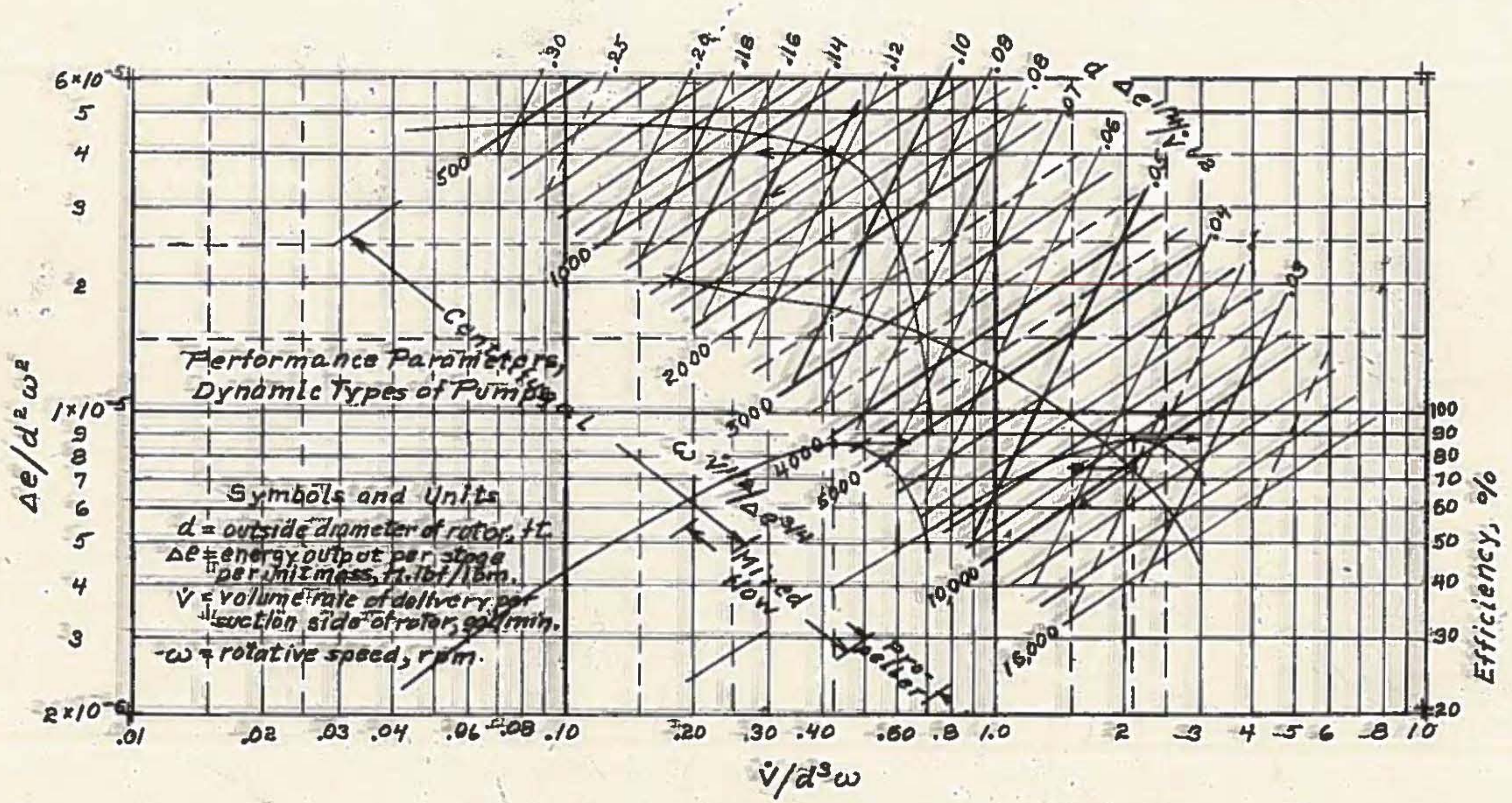

Performance Parameters,
Dynamic Types of Pumps
Centrifugal
Mixed flow
Propeller
Symbols and Units
d = outside diameter of rotor, ft.
Δe = energy output per stage per unit mass, ft.lbf/lbm.
V̇ = volume rate of delivery per suction side of rotor, gal/min.
ω = rotative speed, rpm.
$d\,\Delta e^{1/4}/\dot{V}^{1/2}$
$\omega\,\dot{V}^{1/2}/\Delta e^{3/4}$
$\Delta e/d^2\omega^2$
$\dot{V}/d^3\omega$
Efficiency, %

Fig. 10-7.

A very useful supplementary parameter associates the three variables which in operation are particularly significant; i.e., the rate of delivery, total head delivered and the rotative speed. It becomes

$$\pi_{4}, = \pi_2^{1/2}/\pi_3^{3/4}, = \omega\,(\dot{V}^{1/2}/\Delta e^{3/4}) \qquad (10\text{-}7)$$

When the head delivered per stage is expressed in "feet" (i.e., ft lbf/lbm) and the speed and capacity in revolutions per minute and gallons per minute, and the magnitudes of these are further ones corresponding jointly to optimum-efficiency operating conditions, this index is known as the representative specific speed of a pump (ω_s).

Due to the absence of a size measure in this index, and if data are available on the representative orders of its magnitude for various types or designs of pumps, it becomes uniquely useful in selecting that type which will serve efficiently when furnishing a required head and capacity and turning at a suitable and available speed.

A second supplementary parameter which will serve to indicate the needed size of pump for serving specified requirements efficiently, after having selected in the above manner the suitable type, is known as the specific diameter and becomes

$$\pi_{5}, = \pi_3^{1/4}/\pi_2^{1/2}, = d\,(\Delta e^{1/4}/\dot{V}^{1/2}) \qquad (10\text{-}8)$$

Scales and (inclined) contour lines for these supplementary parameters are included in the graph of fig. 10-7. For their use recall that a given point on a curve of specific head versus specific capacity also fixes the associated specific magnitudes of speed and diameter; the associated efficiency is that taken from its curve at the same specific capacity. ~~An accompanying tabulation, with sketches, indicates reported orders of magnitude of the specific speed for pumps having impellers of various types and proportions or blade orientation.~~ A following example illustrates the use of the figure.

General considerations becoming evident from fig. 10-7 are that -

(a) for operation of an individual pump at a variety of speeds but in such manner that a constant (and preferably) optimum efficiency may be secured, and thus that $\Delta e/d^2\omega^2$ and $\dot{V}/d^3\omega$ continue unchanged, this may be accomplished only under imposed conditions such that the delivery rate is proportional to the speed and the head to its square, with the concurrently requisite power input thus being proportional to the cube of the speed.

(b) among a series of geometrically similar pumps of different size but driven at the same speed and under imposed conditions such that the same (optimum) efficiency may be obtained, the head at which this is obtainable is proportional to square of the size and the volume-rate of delivery to its cube, with a correspondingly requisite power input proportional to the fifth power of the size.

Example 10-1. A given pumping service will require the delivery of water at about 1000 gpm and through a pressure range of about 110 psi (i.e., Δe = 110x144/62.4 = 254 ft.lbf/lbm, or "feet"), and it is desired to drive the pump with an a.c. motor turning at 1760 rpm.

(a) Will any of the (single-stage) pumps for which generalized characteristic curves appear in fig. 10-7 operate in this service with favorable efficiency and, if so, a pump having what diameter of impeller would be suitable ?

(b) At the above loading what requisite power input may be anticipated ?

(c) Delivery through what pressure range would necessarily be imposed on the above pump in order to reduce its delivery rate to 600 gpm, the rotative speed remaining the same, and what power input would be required at this operating condition ?

Solution. (a) The specific speed corresponding to the service as stated above is $1760(1000)^{1/2}/(254)^{3/4} = 1760(31.6/63.7) = 873$. This is within the range for which centrifugal pumps are suitable, but the particular design of pump for which performance characteristics are indicated in fig. 10-7 could evidently serve only quite inefficiently. However it may be seen herewith that a two-stage pump of this design would serve at an optimum efficiency.

That is; at a Δe per stage of 254/2 or 127 ft.lbf/lbm, ω_{sp} is 1760(31.6/37.9) or 1467. Also, at that point on the specific-head vs specific-specific curve which for the pump corresponds to this magnitude of $\omega \dot{V}^{1/2} / e^{3/4}$, $\dot{V}/d^3\omega$ is about 0.48 and the associated efficiency is about the maximum, or 0.85. The correspondingly suitable impeller diameter is thus such that d^3 is about 1000/(.48x 1760) or 1.19, and d = 1.06 ft or 12 3/4 inches.

(b) Power required = (254x1000x8.33)/(33,000x.85) = 75 hp.

(c) For the two-stage pump when delivering 600 gpm, but still operating at 1760 rpm, $\dot{V}/d^3\omega$ = 0.48x(600/1000) = 0.288. From the corresponding point on the characteristic curve; efficiency = 0.74, and $\Delta e/d^2\omega^2 = 4.4\text{x}10^{-5}$ or Δe per stage = $1.06^2 \times 1760^2 \times (4.4\text{x}10^{-5})$ = 153 ft.lbf/lbm and 306 ft.lbf/lbm for the unit, or Δp = 133 psi. Thus, power input = (306x600x8.33)/(33,000x0.74) = 63 hp.

The accompanying table provides a schedule indicating suitable manners of the entry to and use of fig. 10-7 for obtaining desired items when various data and requirements are specified.

Table 10-1. USE SCHEDULE, FIG. 10-7

Case	Stated data or requirements	Desired items	Procedure
1	Pump type, $\dot{V}$ and d	Δe and ω for best eff.	From optimum point on pertinent efficiency curve read $\dot{V}_{sp}$ and compute ω; from corresponding point of head curve read Δe_{sp} and compute Δe.
2	Pump type, $\dot{V}$, d and ω	eff. and Δe	At computed value of $\dot{V}_{sp}$ pass to effic. and Δe_{sp}-curves; read effic. and compute Δe.
3	Pump type, $\dot{V}$ and Δe	d and ω for best effic.	From optimum eff. point pass to Δe_{sp}-curve, read corresponding d_{sp} and ω_{sp} values, compute d and ω.

4	Pump type, $\dot{V}$, Δe and d	ω and eff.	Compute d_{sp}; at intersection with Δe_{sp}-curve read $\dot{V}_{sp}$, Δe_{sp} or ω_{sp} and compute ω; from intersection pass to eff. curve
5	Pump type, $\dot{V}$, Δe and ω	d and eff.	Compute ω_{sp}; at intersection with Δe_{sp}-curve read $\dot{V}_{sp}$, Δe_{sp} or d_{sp} and compute d; from intersection pass to eff. curve
6	$\dot{V}$, Δe and ω	Pump type, number of stages and d, for best eff.	Compute ω_{sp} and, from survey of curves drawn to coordinates of fig. 10-7 for a variety of pump types, (single-stage), note if for any the $\dot{V}_{sp}$-lines through a favorable eff. point intersects suitably the ω_{sp}-contour; if so proceed as in ex. 10-1. Otherwise divide Δe by such integer (i.e., 2, 3, . ., denoting number of stages) as would indicate compliance with requirements, and proceed as before.
7	$\dot{V}$, Δe and d	Pump type, number of stages and ω, for best eff.	Proceed as in case 6, except for initial computation of d_{sp} and use of this parameter as the determining one.

Correlations and procedures such as the foregoing become particularly useful when highly efficient pumps of very large capacity are to be provided and the preliminary construction and testing of a model of the prototype pump is thus justified, as with the pumps having capacities in excess of 10^5 gpm and required heads in the hundreds of feet which are encountered with many projects in our western states. However here, as well as in other instances, it becomes necessary to recognize an appreciable lessening of the attainable efficiency with decrease in pump size. This is due, jointly, to considerations such as (a) a greater relative roughness (art. 6-1 and fig. 6-3) of the surfaces of smaller pumps which in other respects are geometrically similar, (b) disproportionate back-leakage from discharge to entry zones of an impeller, about its exterior and through necessary clearances between it and its casing, and (c) disproportionate frictional losses in bearings et cetera. A relation proposed by Moody for accounting these influences is

$$\frac{1-\eta_p}{1-\eta_m} = (d_m/d_p)^{0.2} \text{ (or less for very smooth model)}$$

To illustrate, a pump (or model) having one third the impeller diameter of a larger one for which the optimum efficiency is 0.85, but geometrically similar, would correspondingly exhibit an efficiency of $[1 - (1 - .85)(3^{0.2})] = 1 - .15 \times 1.246 =]0.813$. *

(* — A somewhat related consideration pertains to a convenient and quite suitable manufacturing practice in which a single size and form of casing is adapted to moderately differing head requirements by the use of impellers which are identical except in outside diameter. Such a departure does introduce, however, a geometric dissimilarity producing characteristic performance curves which, when drawn to coordinates such as those of fig. 10-7, will differ more particularly in the specific capacity at which optimum efficiency is found.)

10-5. Cavitation, Pumps. When for any reason the absolute pressure at entry to a pump or its impeller is so reduced that it approaches the pressure at which vapor bubbles or ones of dissolved gases may be released from the entering liquid stream, the performance of the pump tends to deteriorate quite materially or it may further become incapable of delivery. In this situation the pump is said to be in a state of cavitation.

Except as influenced by the presence of dissolved gases, this pressure is a function of that at which the liquid will normally vaporize from a free surface at the existing temperature, or the so-called saturation pressure $p_{sat.T}$, such as about .36 psia at 70° F for fresh water or 1.5 psia at 115° F. Due to the lessr pressures accompanying the greater relative velocities along the back of an advancing blade, and minimum magnitudes at the entry section of an impeller cell, bubble formation is initiated in that general region. If the bubbles escape from the blade surface, or if they collapse abruptly at the surface on advance of the stream through the cell to higher-pressure zones, an accompanying incessant bombardment of the surface by inrush of the returning liquid causes noisy operation and perhaps vibration, and if persisting produces roughening and pitting of the blade surface through a fatigue-failure at weaker or more severely punished spots. The impeller becomes incapable of inducting and delivering the liquid if the rate of bubble formation is sufficient that they may coalesce and blanket the blade surface.

Through the energy equation the mean absolute pressure in the stream approaching the impeller (p_1) is expressible as, in consistent units,

$$p_1 = p_0 + \rho\left[g(z_0 - z_1) - u_1^2/2 - {}_0\phi_1\right] \qquad (10\text{-}9)$$

where p_0 = absolute pressure at the surface of the liquid in the container from which it is being withdrawn, and atmospheric for an open container;

$z_0 - z_1$ = distance the impeller is submerged below that surface, but becoming subtractive if located above that surface;

$u_1^2/2$ = kinetic energy acquired en route from liquid surface;

${}_0\phi_1$ = energy dissipated en route by fluid friction and turbulence in piping, valves et cetera.

The amount by which the above pressure (p_i) exceeds that minimum actually encountered on the blade surface, and also its excess over $p_{sat,T}$, but ultimately the difference $p_{min} - p_{sat,T}$, are the items primarily affecting the probability of cavitation. The first excess, or $p_i - p_{min}$, depends on items such as the rotative speed and size of the impeller, the angular spacing of the blades and their ~~spacing,~~ number the pressure gradients both along and across the streams through the cells ~~or~~ and the energy dissipation associated with entry of the streams into the cells. However this difference is not readily expressible.

In this situation and again from the concept of dynamically similar flow through geometrically similar channels, when or if any pair of the foregoing "specific" parameters have unique simultaneous magnitudes, one may reasonably conclude that in such circumstances the pressure distributions in similar propellers with further be similar, and that there is therefore a parallel ability to anticipate through dynamic-similarity considerations that minimum magnitude of $p_i - p_{sat,T}$ at which cavitation may still be avoided. A consequent current practice, which is validated by tests and experience, is that of ascribing to pumps of given character a maximum safe magnitude of a parameter which is described as a suction specific speed and is in principle defined as

$$\text{Suction specific speed, } S, = \frac{\omega \; \dot{V}^{1/2}}{[(p_i - p_{sat,T})/\rho]^{3/4}} \quad * \qquad (10\text{-}10)$$

Note in this connection that the ratio $(\omega_{sp}/S)^{4/3}$ thus equals $\dfrac{p_i - p_{sat,T}}{\rho \, \Delta e}$.

There is evidence that, in the units of fig. 10-7 and for the normal centrifugal pump. cavitation is avoided at rated capacity by values of S of about 8000, but possibly 50% or more greater for larger pumps of very careful design and construction. Through the use of such values one may evidently anticipate suitable maximum magnitudes of ω and/or $\dot{V}$ for a general

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(* - In the literature relating to this specific speed, and also to the parameter of equation 10-11a, it is customary to describe the quantity $(p_i - p_{sat,T})/\rho$ as a "head", or specifically a "net positive suction head" expressed in "feet of the fluid". This terminology, originating in the considerations of eq. 2-1a or 2-1b and referred to again in eq. 5-4, is numerically permissible when the inconsistent pound-force and pound-mass units are employed. However it is so very liable to introduce dimensional and other confusions that it is again preferred to avoid such usage.)

character of pump when a given value of $p_i - p_{sat,T}$ is encountered, or the converse.

Another manner of procedure for the avoidance of cavitation employs a dimensionless parameter which is known as a <u>cavitation index</u> (or <u>number</u>) and, for pumps, is in principle defined as

$$\text{Cavitation index, } \sigma \text{, for pumps,} = (p_i - p_{sat,T})/(\rho \Delta e) \quad (10\text{-}11a)$$

Minimum safe magnitudes of this index may then be ascribed to various pumps. However, by referring to the observation immediately following equation 10-10, it is seen that, at a representative general magnitude of S and for a pump type having a representative magnitude of ω_{sp}, a correspondingly suitable minimum (or critical) magnitude of σ may alternatively and usefully be expressed in the form

$$\sigma_{crit}\text{, for pumps,} = (\omega_{sp}/S)^{4/3} \qquad (10\text{-}11b)$$

To illustrate the use of foregoing relations, for a pump suitably meeting the requirements of example 10-1 and for which the data prescribed a specific speed of 940, a suitable maximum value of σ of about $(940/8000)^{4/3}$ or 0.057 is thus indicated. If the pump is to induct fresh water at 70°F (i.e., p_{sat} = 0.36 psia or 52 lbf/sq.ft abs. and ρ = 62.3 lbm/cu.ft or 1.94 slugs/cu.ft) from an open tank at about sea level, equation 10-11a indicates that the minimum magnitude of p_i for which cavitation would be avoided is about (.057x62.3x231 + 52 =) 872 lbf/sq.ft. By equation 10-9, and if the kinetic energy increase and energy dissipation in the intake system from tank to pump were presumed to be relatively neglible, the maximum elevation <u>above</u> the water surface at which the pump might suitably be located becomes about [(2117-872)/62.3=] 20 feet.

However the neglected items may be quite appreciable and necessitate a less elevation. Such would also be necessary if it were desired that the pump might provide a greater delivery rate if or when required, doing so without cavitation and whether operating at the same rotative speed but providing a reduced head or operated at a speed increased sufficiently to provide the same head. For either option the <u>actual</u> specific speed at which the pump need be operated is greater, with consequent increase of σ and of the requisite p_i.

10-5. <u>Inter-relation of Pump and Delivery-system Characteristics</u>. When investigating the suitability of an available or proposed pump for serving a given delivery system, it is essential to know both the performance characteristics of the pump and the characteristics of the system. By the latter is meant the relation between the desired delivery rates and the pressures (at pump discharge) required for obtaining such rates. Only by knowledge of both may one judge whether they are suitably compatible or can be made reasonably so by proper pump control, or may anticipate the penalties which will otherwise be incurred by excessive energy dissipations and associated excess power costs.

Delivery systems may range in character from ones in which the requisite pressure at entry is determined primarily by the energy dissipation due to fluid friction and turbulence, to ones in which a more or less constant supply pressure must be maintained and any pressure drop en route is relatively minor. The first situation is illustrated by a long pipe line which discharges to the atmosphere and for which the elevations at entry and exit differ negligibly. It will be recalled that, in such situation and for flow at the typically higher magnitudes of the Reynolds index, the energy dissipation and consequent requisite supply pressure are nearly proportional to the square of the volume-rate of through-flow (i.e., in Fig. 6-3, ϕ/u^2 or ϕ/G^2 are about constant). The situation is represented in Fig. 10-8 (to linear coordinates of ϕ and $\dot{V}$) by the line curving upwardly to the right.

Relative to associated pump characteristics, the curves of "head" and efficiency to a common abscissa of $\dot{V}$ for delivery system and pump, when the latter is for example operated at speeds ω_1, $1\frac{1}{2}\omega_1$ and $2\omega_1$, illustrate features noted in the foregoing material. A pump so selected that it exhibits a favorable efficiency when operated at a speed such as to provide the "head" required by the delivery system at one rate of through-flow, will evidently continue to do so at other rates if operated at speeds proportional to those rates.

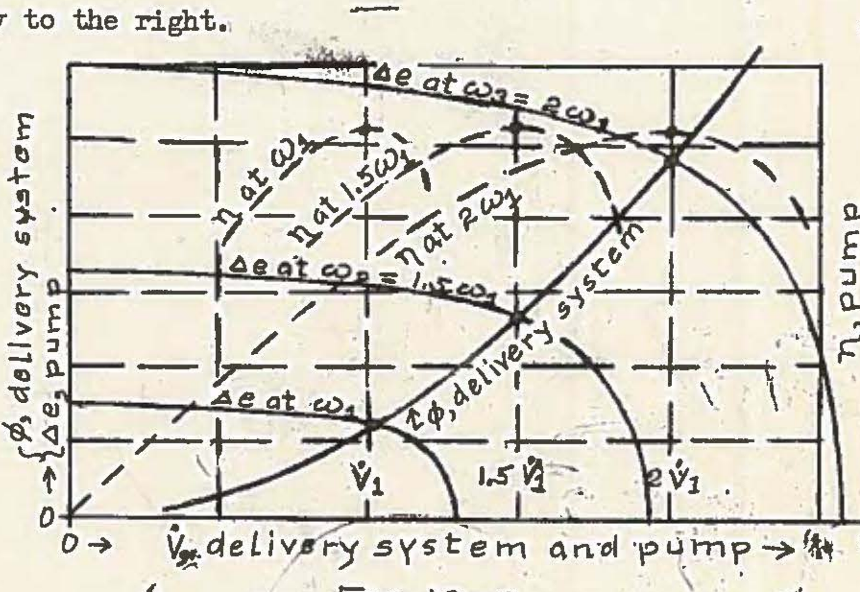


Fig. 10-8

The second situation, in which a fairly constant exit pressure from a delivery system is to be maintained and the energy dissipation in the system is relatively moderate, is illustrated by a pipe line delivering to a tank at a considerable elevation above the pump by which it is served, or to the drum of a boiler. The required pressure at the discharge-end of the line (p_2) and the necessary pressure at pump discharge (p_1) may be somewhat as indicated in Fig. 10-9. The figure represents also characteristic "head" and efficiency curves for a centrifugal pump which at suitable speed and a desired delivery rate ($\dot{V}_a$) will give a sufficient discharge pressure and a favorable efficiency.

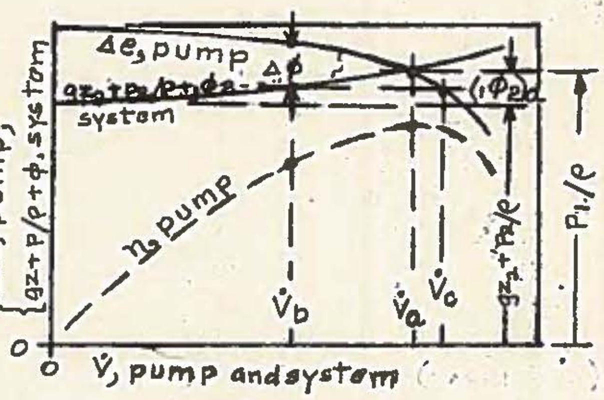


Fig. 10-9

If the pump speed is necessarily about constant, as when driven by an a-c motor, a suitably selected pump will meet these particular requirements but be incapable of providing higher rates of delivery to the system. If a lower rate is desired ($\dot{V}_b$), it is evident that a supplementary energy dissipation ($\Delta\phi$) must be introduced, as by partial closure of a valve in the delivery line, and that this will further be accompanied by a reduction in pump efficiency. An alternative procedure would be to permit the pump to deliver at the greater rate ($\dot{V}_c$) which, for it, corresponds to the lower pressure required by the delivery system, but to return (or "bypass") the excess delivery back to the pump intake through a suitably opened valve. Excess energy dissipation is introduced by either procedure.

Better compatibility between pump and delivery-system characteristics may be obtained in situations such as the above by the provision of several like but lower capacity pumps, so installed as to enable operation of one at lowest requisite delivery rates but of two or more (in parallel) at greater rates. When otherwise adaptable, a positive-displacement pump (Art. 10-1) capable of providing varied delivery pressures even at a single operating speed may serve such situations effectively and with better overall efficiency.

Delivery systems between the above extremes and of varied character may frequently be encountered, but the foregoing may have indicated sufficiently the considerations which will continue to be relevant.

10-6. Impulse Type, Hydraulic Turbine. Except as accomplished rather inefficiently by the early water wheel, actuated by the momentary deceleration of water falling onto its blades, the turbine described as of impulse type was a pioneer in the transformation to shaft-work of the energy stored in an elevated and open body of water which represents accumulated rain or snow fall. In the impulse-turbine installation the transformation is a progressive one from the stored geopotential energy

(a) to the geopotential, kinetic and flow-work energies of the water entering the penstock through which it passes to the turbine;

(b) to increasing flow-work and decreasing geopotential energies accompanying the downward passage of the water through the penstock;

(c) to the kinetic energy of one or more jets generated by passage thence through one or more suitably directed nozzles, into a turbine casing in which the pressure is again atmospheric; and ultimately

(d) to shaft-work, through the agency of "buckets" which are mounted on the periphery of the turbine runner (i.e., rotor) and act to divert and decelerat the stream leaving a nozzle.

The transformations are inescapably accompanied by some energy dissipation through fluid friction and turbulence, and also by the discard of some residual geopotential and kinetic energy in the departing water, but the efficiencies obtained may be very favorable.

The general arrangement of the turbine known also as the Pelton wheel and the character of the buckets is indicated in Fig. 10-10; also the general character of the velocity distribution in the absolute path of a stream in its passage along the face of a retreating bucket. The number and spacing of the buckets is such that a diverting of

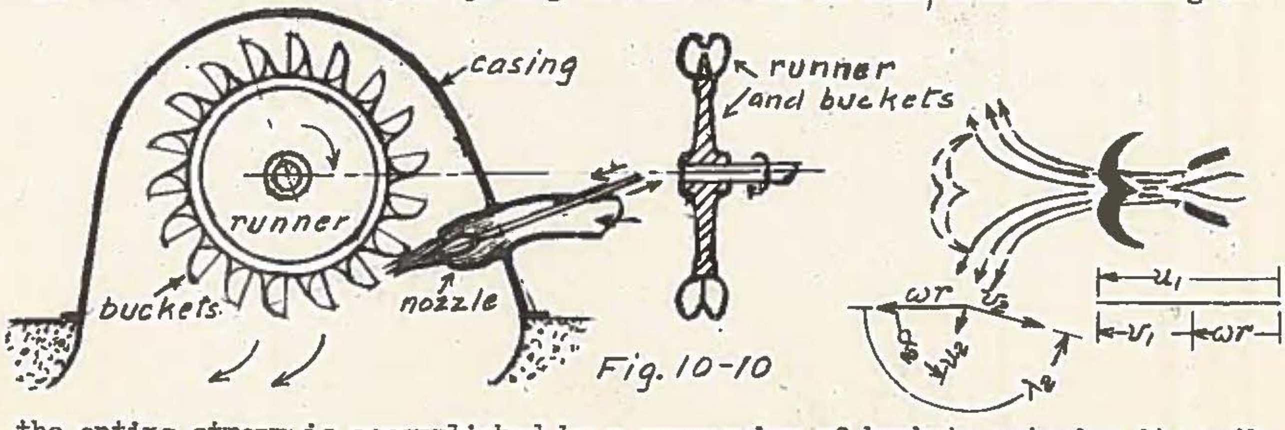


Fig. 10-10

the entire stream is accomplished by a succession of buckets entering its path.

More specifically, the energy transformation from kinetic energy to flow-work by a bucket is effected through the agency of the characteristic velocity and pressure distributions across a stream the path of which is forcibly diverted, by the intrusion of the bucket and of a character comparable to that exhibited in the forced (spiral) vortex of Fig. 9-9. The aggregate of the tangential components of the forces, attributable to the greater-than-atmospheric pressures along the face of the bucket which is diverting the stream, furnishes the force and torque effecting the motion of the runner against a "load" extraneously imposed through the turbine shaft.

As analyses are impracticable whereby one may anticipate the velocity (and pressure) distributions within the stream while being diverted, and thereby the resultant forces, again representative mean velocities are customarily ascribed to the arriving and departing stream, and the more significant of associated force components are expressed through facilities such as equation 9-19a. Performance expectations as so derived are quite informative. The vector diagrams accompanying Fig. 10-10 indicate these and influential related velocities. These are -

u_1 = absolute velocity of arrival of the jet, in a direction tangential to the path of the buckets;

ωr = tangential velocity of the retreating bucket, at a mean radius r;

v_1 = relative velocity at which the stream arrives at the moving bucket, and equal to $u_1 - \omega r$;

v_2 = relative velocity at which the stream departs from the exit edge of bucket, of a magnitude taken to be moderately less than v_1 (or $= kv_1$) due to all such influences as may impede its passage along the bucket surface, and in a direction relative to the (moving) bucket which is taken to be parallel to the ~~blade~~ bucket surface at that point;

u_2 = absolute velocity of departure of the stream and the resultant, in magnitude and direction, of above velocities ωr and v_2.

There is evidence that k may be regarded as in the order of 0.85.

In conformity with eq. 9-19a, except as the terms involving p_1 and p become irrelevant since the pressure in the entire environment surrounding the stream and bucket is atmospheric, the force component F_x acting on the face of the bucket in its direction of advance may be written as

$$\begin{aligned} F_x/\dot{m} &= u_1 - u_2 \cos\alpha_2 \quad \text{or, as } u_2\cos\alpha_2 = \omega r + v_2 \cos\lambda_2, \\ &= (u_1 - \omega r) - k(u_1 - \omega r)\cos\lambda_2 \end{aligned} \tag{10-12a}$$

Or in terms of a dimensionless "force coefficient",

$$\begin{aligned} F_x/\dot{m}u_1 &= 1 - \omega r/u_1 - k\cos\lambda_2 + k(\omega r/u_1)\cos\lambda_2 \\ &= (1 - \omega r/u_1)(1 - k\cos\lambda_2) \end{aligned} \tag{10-12b}$$

Note that, as $\cos\lambda_2$ is negative when λ_2 exceeds 90°, $(1 - k\cos\lambda_2)$ would normally exceed unity.

As the product of the force (F_x) on the buckets times their tangential velocity is the power thereby obtainable,

$$P_{dyn}, \text{ or } F_x(\omega r), = (\omega r)(\dot{m}u_1)(1-\omega r/u_1)(1-k\cos\lambda_2)$$

or $$P_{dyn}/\dot{m}\,u_1^2 = (\omega r/u_1)(1-\omega r/u_1)(1-k\cos\lambda_2) \qquad (10\text{-}13)$$

If the rate of "available" energy input to the turbine via the nozzles, or correspondingly its ideal power output, is taken to be the product $\dot{m}\,u_1^2/2$, the efficiency with which this kinetic energy is transformed to work thus becomes

$$\text{Eff. of transformation} = P_{dyn}/(\dot{m}\,u_1^2/2)$$
$$= 2(\omega r/u_1)(1-\omega r/u_1)(1-k\cos\lambda_2)\ * \qquad (10\text{-}14)$$

(* - The actual or net power output via the turbine shaft, and its actual or overall efficiency, are necessarily moderately less than the above due to the slight energy dissipation in passage of the stream through the nozzle, impedance to the turning of the rotor (i.e., a "windage loss") due to its rotation in the spray-laden atmosphere, bearing friction, power diverted for driving speed-governing mechanisms, et cetera.)

Significant considerations indicated by above relations are as follow.

(1) For a runner having buckets of given leaving angle (λ_2), and except as there may be variation of k, the transformation efficiency is a direct function of the bucket speed-ratio ($\omega r/u_1$) only.

(2) By equating to zero the derivative $d\eta/d(\omega r)$, it is seen that, again except as k may vary, the efficiency is maximum when $\omega r/u_1$ is 0.5, and correspondingly that

$$\text{Max efficiency} = 0.5(1-k\cos\lambda_2).$$

This might attain unity only if k were 1.0 and λ_2 were 180°, but of necessity this is sufficiently less than 180° that the stream leaving a bucket does not impinge on the back of the following one.

(3) For a runner with buckets having a given angle λ_2 and negligibly varying k, and if the turbine is so governed as to maintain a given and preferably the above optimum value of $\omega r/u_1$, the power output is directly proportional to the mass-rate of flow through the nozzle. Thus speed-governing of the impulse turbine is most readily effected through change of that rate, by varying the exit area of the nozzle. This is done by adjusting the axial position of a tapered "needle" which is within and concentric with the nozzle, as in Fig. 10-10.

(4) At a given mass-rate and velocity of the jet, force F_x and the consequent torque of the runner increases directly with decrease of speed-ratio $\omega r/u_1$. But constant speed is normally desired, and a torque increase gained by a ratio less than about 0.5 would be at the expense of a reduced efficiency.

The force (F_y) on a bucket in direction normal to its plane of motion may similarly be expressed as $F_y/\dot{m}\,u_1 = k(1-\omega r/u\ )\sin\lambda_2$. But the indicated symmetric split of the jet (fig. 10-10) by the bucket gives two equal but oppositely-directed forces and thus zero axial thrust on the runner.

10-7. <u>Reaction Turbine</u>. The arrangement described as the reaction, or <u>Francis</u>, type of hydraulic turbine differs more particularly from the impulse type in the features that the arriving water fills the casing and the cells formed by the rotor blading, and that the energy provided by the water is transformed to shaft-work wholly by strategic diverting of the streams passing through the cells, without the intervening step of conversion to kinetic energy by a nozzle and instead with progressive pressure drop en route through the cells. In some respects the arrangement resembles more the centrifugal pump.

But, as the objectives of pump and turbine are opposite, in details of form and otherwise they differ distinctly. For ones in which the flow through the rotor is negligibly axial in direction and their directions of rotation are both clockwise, fig. 10-11 illustrates features in which the casing and blading and the velocity distribution in individual cells will necessarily differ. As in the reaction turbine better efficiency may be maintained at various loads through guidance of the streams approaching the rotor by vanes, or <u>wicket gates</u>, capable of variable orientation, such are indicated for the turbine.

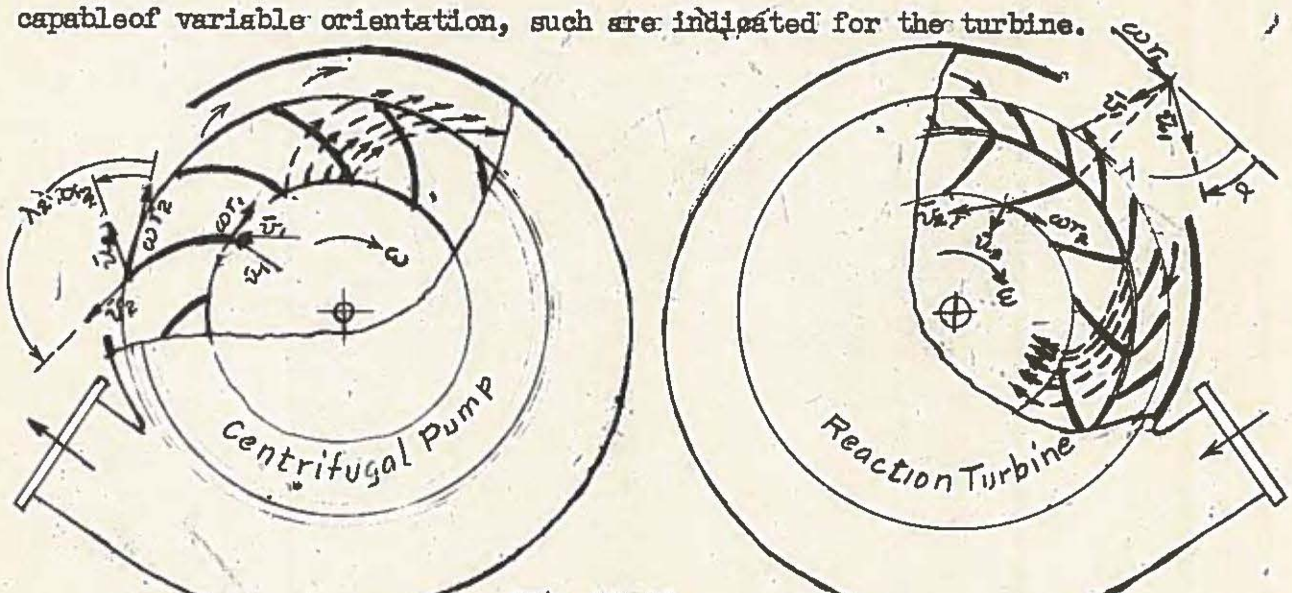


Fig. 10-11

Again with such propriety as the stream velocities at entry and exit sections of the rotor cells might be represented by single vectors, as in figures 10-1 and 10-11, the torque and power deliverable by the rotor might again be expressed by equations 10-2 and 10-3 if angles λ are ones as indicated for the turbine. The algebraically negative results suitably denote the energy <u>departure from</u> the rotor as shaft-work. But the blading of the reaction turbine is more commonly such as to cause a flow configuration of <u>forced-spiral</u>

vortex nature only upon entering a rotor cell, their surface orientations thereafter becoming progressively such as to cause the leaving streams to have more nearly a forced-spiral vortex configuration. Fig. 10-12 pictures a typical blading conformation for such a runner. For this those equations would be inapplicable.

For such reasons, and except as one is unquestionably aided by an awareness of basic hydrodynamic considerations, the design and the anticipation of performance characteristics of the reaction turbine runner are again as much an art as a science.

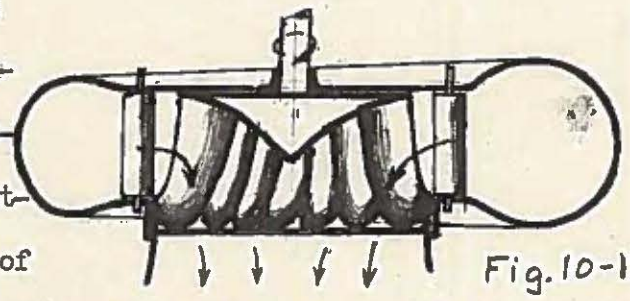


Fig. 10-12

However items which are more determinable, and of direct concern to both the purchaser and the supplier of a turbine, include ones such as -

(a) The maintainable difference between the elevation of the free surface in the reservoir in which the water may be accumulated, from prior or current rain (or snow) fall in nearby and higher regions, and the minimum elevation at which the turbine may be located while still permitting unimpeded off-flow of the leaving water;

(b) The volumetric and mass rate at which the water may be permitted to pass through the turbine without undesirable lowering of the water level in the reservoir;

(c) The corresponding power output (P_{out}) which the turbine may normally be required to furnish, both ideal and actual and while complying with the foregoing limitations;

(d) The efficiency with which the energy transformation is accomplished in and by the turbine.*

* Note in these connections that the consideration of paragraph (d) of page 201, relating to the pump, is not relevant with the turbine. Here the available "head" is established independently, and also in its operation the rate of through-flow of the water is independently controlled in accord with the desired power output and the required rotative speed, such as that of the a.c. generator by which it frequently is loaded.

For effectively steady flow conditions these items and related ones may be associated to advantage through the following composite energy equation, relating to the entire installation of the reservoir, the piping constituting the penstock through which the water passes from reservoir to turbine, the turbine, and the channel forming the draft-tube through which the water passes from the turbine to the tail-race into which it is discarded. Positions identified by the several subscripts are ones as indicated in fig. 10-13.

Symbol p denotes absolute pressures, but elevations are ones relative to that at the surface of the stream leaving in the tail-race. The units are to be regarded as consistent ones. The energy equation becomes -

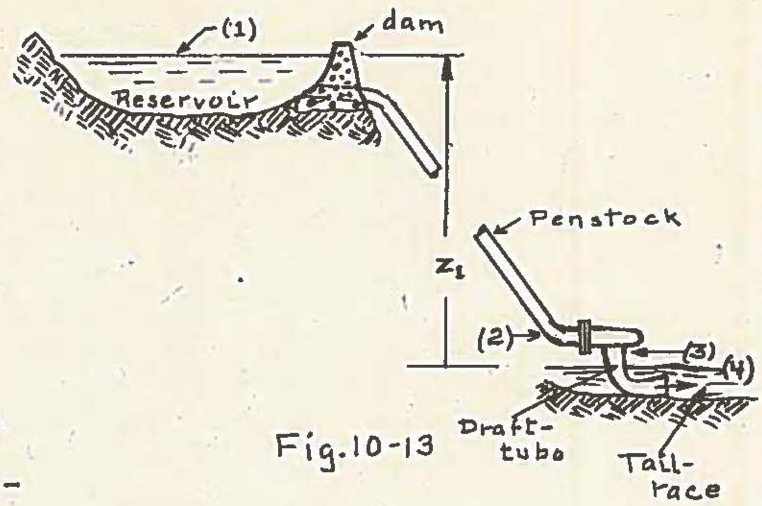


Fig.10-13

$$gz_1 + p_{atm}/\rho = gz_2 + p_2/\rho + u_2^2/2 + {}_1\phi_2 ,$$

$$= gz_3 + p_3/\rho + u_3^2/2 + {}_1\phi_2 + P_{out}/\dot{m} + {}_2\phi_3 ,$$

$$= p_{atm}/\rho + u_4^2/2 + {}_1\phi_2 + P_{out}/\dot{m} + {}_2\phi_3 + {}_3\phi_4 \qquad \text{10-15}$$

It is seen from these relations that, even for an installation in which p_{atm}, z_1, z_2, and z_3 are independently established items, and all components of the system are of stated character, the pressure existing at pertinent points in the system will necessarily vary with the power output and the flow-rate required for obtaining that output.

Energy equations relating to individual components of the system may be written directly from the above. Ones of direct present interest are -

(1) For the penstock; $$(p_2 - p_{atm})/\rho = g(z_1 - z_2) - u_2^2/2 - {}_1\phi_2 , \qquad \text{10-16}$$
where ${}_1\phi_2$ denotes the energy dissipation due to fluid friction and turbulence in passage of the water to and through the penstock piping and the required valves. An unsteady-flow which is also of much practical concern is one in which, due to abrupt decrease or loss of load on the turbine and the associated necessity for like stoppage of inflow to it, the inertia of the arriving mass of water in the penstock may cause excessive and possibly damaging pressures in portions nearer the turbine, producing what is known as water hammer. These may be moderated by providing for a more gradual stoppage, by diverting the arriving water to a suitable storage region or directly to the tail-race. Analyses relating to this situation are beyond the scope of this material.

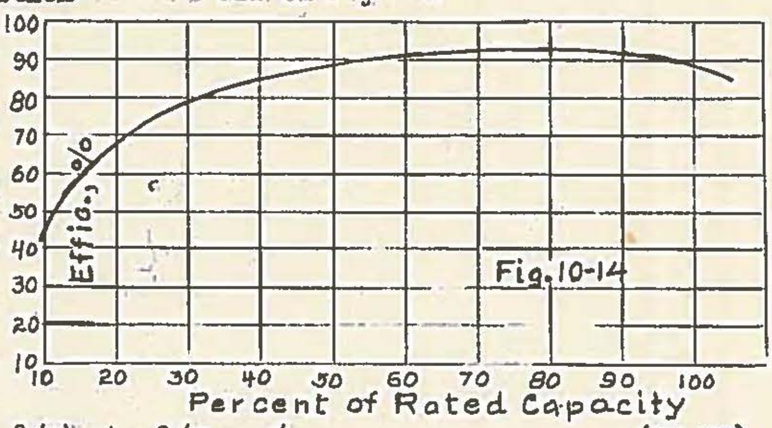
(b) <u>For the turbine</u> -

$$P_{out}/\dot{m} = \left[g(z_2 - z_3) + (p_2 - p_3)/\rho + u_2^2/2\right] - u_3^2/2 - {}_2\phi_3 \qquad (10\text{-}17)$$

where the bracketted term may be taken to express the energy input per unit mass which is ideally available for transformation by the turbine to shaft-work. Correspondingly,

$$\text{Efficiency } (\eta) \text{ of turbine} = \frac{P_{out}/\dot{m}}{g(z_2 - z_3) + (p_2 - p_3)/\rho + u_2^2/2} \qquad (10\text{-}18)$$

Magnitudes and the manner of variation of this efficiency for a typical reaction turbine are indicated in fig. 10-14, to an abscissa of the power output relative to that for which it is "rated".



(c) <u>For the draft-tube</u> -

$$p_3/\rho = p_{atm}/\rho - (gz_3 + u_3^2/2) + u_4^2/2 + {}_3\phi_4 \qquad (10\text{-}19)$$

In its expressing of the pressure at the exit from the turbine runner this relation has direct significance in connection with the influence of that on the occurrence or avoidance of cavitation in the rotor cells.

As the turbine <u>and</u> draft-tube are so intimately related an energy account for this combination, regarded as a unit, is informative. By introducing in eq. 10-17 the above evaluation of p_3/ρ,

$$P_{out}/\dot{m} = \left[gz_2 + (p_2 - p_{atm})/\rho + u_2^2/2\right] - u_4^2/2 - {}_2\phi_3 - {}_3\phi_4 \qquad (10\text{-}20)$$

also, $\quad \text{Efficiency, turbine plus draft-tube,} = \dfrac{P_{out}/m}{gz \; - (p - p_{atm})/ \; - u \;\; /2} \qquad (10\text{-}21)$

A suitable index of the performance of the entire system is

$$\text{Overall efficiency of installation} = (P_{out})/\dot{m})/\, gz_1 \qquad (10\text{-}22)$$

These relations apply directly or are readily adaptable to an existing situation whether the turbine is of impulse, reaction or propeller type.

10-8. <u>Propeller-type Turbine</u>. The propeller-type of turbine is again the opposite in purpose to that of the propeller-type pump of Art. 10-3, and also in the general angular orientation of its radial blades, or foils, relative to a plane normal to the turbine shaft. Figure 10-15 indicates its general form and arrangement. In a manner similar to the reaction turbine, the volumetric flow-rate and the angular orientation of the stream entering the propeller housing are governed by the angular positioning of a ring of wicket gates. In turbines known as the <u>Kaplan</u> type the angular orientation of the blades is further governed by and in conformity with the positioning of the wicket gates. Or the propeller is said to have <u>adjustable pitch</u>, the term pitch having a significance comparable to that of a screw thread. For reasons similar to ones noted below, this procedure is conducive to the maintenance of a favorable turbine efficiency over a much greater range of loads than is obtainable with blading not having pitch control.

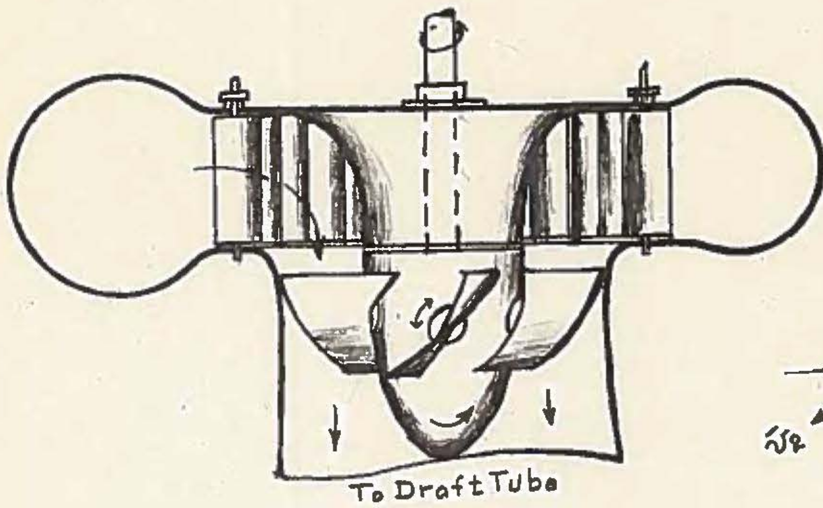


Fig. 10-15

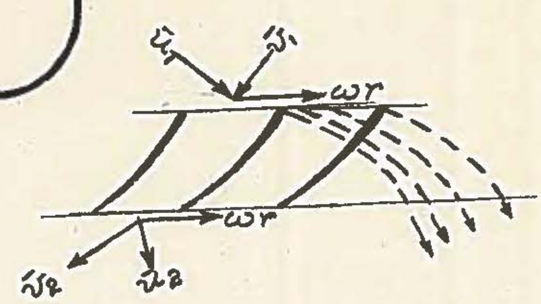


Fig. 10-16

Again to such extent as one may represent by single vectors the velocities of the individual streams entering and leaving the channels formed by neighboring pairs of blades, fig. 10-16 represents these and related velocities as pertaining to that annulus of the entire stream which is at radial distance <u>r</u> from the center-line of the runner. The figure also suggests the developed absolute path followed by the stream in passage at this radial location through the channel formed by the (moving) blades, and the velocity distribution in that stream.

As the tangential velocity of a point on the blade (ωr) evidently varies directly with the radial position of the point, and as further the radial distribution of the velocities in the stream approaching the plane of the runner (v_1) probably resembles that of the free helical vortex (Art. 9-9a), the angular orientation of the blade surfaces ~~are~~ correspondingly so ~~varied~~ differ <u>from hub to tip</u> that throughout the stream may enter the individual channels with minimum disturbance and leave with minimum residual velocity. The blades are thus of <u>variable pitch</u>, in distinction to the above <u>adjustable</u> pitch of the entire blade of the Kaplan runner.

Curves of efficiency versus percentage of rated power capacity for the propeller type of hydraulic turbine are typically about as indicated in fig. 10-17 for (a) ones with fixed blades and (b) ones with blades of automatically variable pitch. The following article will indicate that either is best adapted to installations having available only a quite moderate "head" of water.

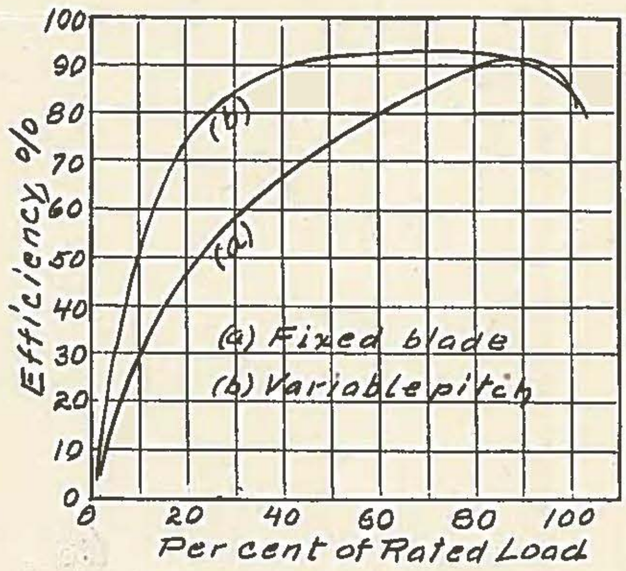


Fig. 10-17

10-9. <u>Generalized Performance Correlation, Hydraulic Turbine</u>. The data which the turbine purchaser may furnish most readily to the supplier, and which are for that reason advantageously associated in a general performance index, include (a) The "head" made available by the topography of the region in which the turbine will be located; or, more significantly, the energy per unit mass of water arriving at the turbine (Δe) but, when evaluated in ft.lbf/lbm, <u>equal numerically</u> to the "head in feet".

(b) The probable rate at which the water will collect in the reservoir and the suitable rate of withdrawal ($\dot{V}$), with these and the above "head" acting jointly to fix the possible or suitable power output that may be demanded of the turbine.

(c) A suitable rotative speed, this as influenced perhaps of the requirements of an a.c. generator which the turbine will drive and by mechanical limitations associated with the type of turbine better fitting the general situation.

In this connection recall that the three items Δe, $\dot{V}$ and ω are also the ones associated in the dimensionless specific-speed parameter evolved in Article 10-4 in relation to the pump; i.e., $\omega_s = \omega\, \dot{V}^{1/2}/\Delta e^{3/4}$. But this parameter may be put in terms of the power output (P_{out}) by noting that $P_{out} = \dot{m}\, \Delta e\, \eta$, or $(\dot{V}\rho)\Delta e\, \eta$, so that equivalently

$$\omega_s = \left[\omega \frac{(P_{out})^{1/2}}{(\Delta e)^{5/4}}\right](\rho\eta)^{-1/2}$$

But in turbine terminology it is the practice to define as the specific speed only the bracketted portion of this parameter; i.e.,

$$\omega_{s,\ \text{turbine}} = \omega\,(P_{out})^{1/2}/(\Delta e)^{5/4}, \qquad (10\text{-}23)$$

with evaluation of ω in rpm, Δe in ft.lbf/lbm (or "feet head") and P_{out} in horsepower.

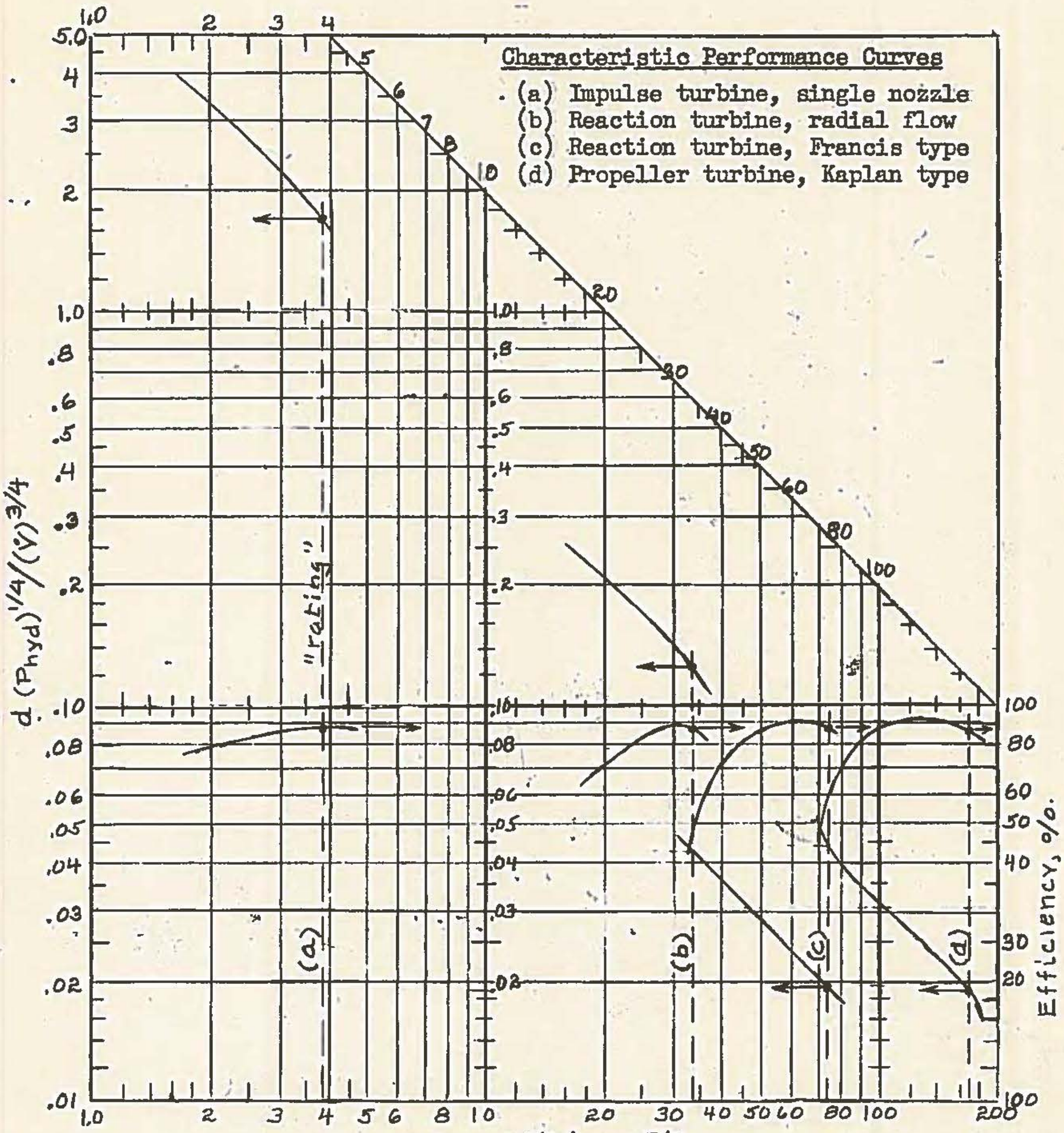
This usage is quite understandable, as within the temperature range normally encountered in practice the density of water changes sufficiently moderately as scarcely to justify its consideration as a variable. However the usage has unquestionably caused persistent dimensional confusions, for the index as so defined "misses" by the product $M^{-1/2} L^{3/2}$ the non-dimensionality which is in principle required to validate its use in generalized performance correlations.

Turbines of a given type and geometric proportioning are customarily regarded as having a correspondingly representative specific speed, this normally being that corresponding to the greatest power output at which a favorable efficiency is still obtainable. The employment of the power output in defining the specific speed is also understandable as this is the item of major concern to the purchaser.

However it is advantageous also to consider an index $\omega (P_{hyd})^{1/2} / (\Delta e)^{5/4}$ $[= \omega_{s,\ turbine} / (\eta)^{-1/2}]$ in which, using the subscript significances of fig. 10-13 and eq. 10-21, $P_{hyd} = \dot{m}\ (gz_2 + p_2/\rho + u_2^2/2)$ in consistent unit; or, in common engineering units, $P_{hyd,\ hp} = \dfrac{\text{lbm/min x energies in ft.lbf/lbm}}{33{,}000}$. This (again dimensional) index is employed as the abscissa of the graph of fig. 10-18, which is comparable in character and utility to that of fig. 10-7 for pumps. The ordinate scale to the right is that of efficiency.

The extended ordinate scale to the left is that for a further index which serves to associate conveniently the hydraulic horsepower input, the requisite rate of through-flow of the water, and the required relative size of the type of turbine which otherwise satisfies the imposed operating conditions. This index is an adaptation of the (dimensionless) specific-diameter parameter of fig. 10-7, or $d\ \Delta e^{1/4} / \dot{V}^{1/2}$. That is, by noting that $\Delta e = P_{hyd}/\dot{V}\rho$, that parameter may be written as $[d\ (P_{hyd})^{1/4} / \dot{V}^{3/4}]\ \rho^{-1/4}$. Again neglecting the trivially variable density (but thus leaving again a dimensional combination), a useful index which evolves may be described as the specific diameter of the turbine and defined as

$$d_{s,\ turbine} = d\ (P_{hyd})^{1/4} / \dot{V}^{3/4} \qquad (10\text{-}24)$$



d = outer diam. of runner, ft. Δe = available energy of entering stream, ft.lbf/lbm or "feet of water".
P_{hyd}, = power input through agency of entering water, hp. $\dot{V}$ = volumetric rate of entry, cu.ft/min.

Fig. 10-18

in which

d = outer diameter of turbine runner, ft;
P_{hyd} = rate of available energy entry via arriving stream; and
$\dot{V}$ = rate of through-flow, cu.ft/min.

To the coordinates of fig. 10-18 there are plotted representative performance curves for a typical impulse type, two reaction types and a propeller hydraulic turbine.* The very low specific speed indicated for the impulse turbine

(* As the power delivered to and by the turbine is varied by change in the positioning of the needle in the nozzle of the impulse turbine, of the wicket gates of the reaction or propeller types of turbine, and further of the blading itself in the Kaplan-type, they may properly be regarded as not maintaining a constant geometrical configuration and so be thought to be incapable of performance correlations by utilization of the concepts of geometric and dynamic similarities. However in principle the technique still retains validity as, at like positioning of the control elements, like simultaneous values of the above indices should be encountered with, for example, model and prototype.

is indicative of its typical greater suitability in mountainous areas, where "heads" in excess of 5,000 feet have been made available and utilized. For the reaction turbine a representative range of specific speeds is from about 15 for ones having primarily a radially-inward through-flow, to about 80 for ones having runners such as pictured in fig. 10-12, with a corresponding better adaptability to hilly rather than mountainous areas. Specific speeds of 100 to 150 characterize the propeller-type turbine, these being correspondingly more suitable for regions where available "heads" are in the order of 10 to 100 feet.

The occurence and consequences of cavitation (Art. 10-5) are quite as much to be avoided with the reaction and propeller types of turbine as with the pump. Also the probability of its occurence with the latter may in general be greater as -

(1) the facilitating of proper maintenance dictates the location of the runner sufficiently above the water level in the tail-race;

(2) the associated less-than-atmospheric pressure at runner exit (if the draft-tube remains filled with water), and the consequent greater pressure drop obtainable en route <u>through the runner</u>, enables the use of a smaller runner and casing for obtaining a given power output, and so puts some premium on a maximum safe elevation.

Due care is thus necessary in the selection of the relative elevation of the runner, and also in the design of the draft-tube. In fact some sacrifice in the vacuum at runner exit ($p_3 - p_{atm}$) may be advantageous in order that a lower loads and flow-rates p_3 may not be so low as to cause cavitation. This may be secured by increase of ${}_3\phi_4$ through the introduction of impedences to the flow through the draft-tube.

The velocity distributions indicated in figures 10-11 and 10-16 indicate that the zone of minimum pressure, and associated maximum cavitation probability, is on the rear face of the advancing blade, and it will be near the exit end of the cell. The degree to which this (minimum) pressure approaches the saturation pressure corresponding to the water temperature is again a major factor influencing the encountering of bubble formation. Due to the inability to anticipate this pressure directly, it is again the practice to regard p_3 and the energy decrease per unit mass of fluid en route through the runner as capable jointly of providing the required evidence whereby cavitation may be avoided. That is, the cavitation index of eq. 10-11a is again employed, except as written in the form

$$\text{Cavitation index } (\sigma), \text{ turbine} = (p_3 - p_{sat,T})/({}_2\Delta e_3) \qquad (10\text{-}25)$$

There is evidence that a minimum safe value of σ for the reaction turbine is in the order of 0.1 but, due to the characteristically moderate values of ${}_2\Delta e_3$ for which it is best adapted, is in the order of about 0.8 for the Kaplan turbine.

10-10. Fluid Coupling. In principle the fluid coupling of this article and the torque converter of the following one are simply ingenious combinations of a power-driven pump runner and a loaded turbine runner which are in such juxtaposition and so encased that a working fluid may and will circulate continuously and rather directly between the two. The fluid is commonly a low-viscosity oil. Both runners may be described roughly as of the reaction type.

Both serve to transmit torque from the pump shaft to the turbine or output shaft through the agency of the dynamic forces associated with a toroidal character of flow-path which the fluid is enforced to take in its circulation through the turning rotors. In this respect they accomplish hydrodynamically the same objective as is done mechanically by a meshing pair of gears. But they do so -

(a) with effective cushioning of shocks and the absorption of torsional vibrations, which with a gear train would be transmitted between the driving and driven shafts; and

(b) with the capability of transmitting the power with a smooth and considerable variation in the speed-ratio as between the rotative speeds of the driving and driven shafts, which might only be approached by continuous gear shifting if a gear train were used.

However, with respect to the latter feature, the simpler coupling is well adapted only to situations in which these speeds are rather nearly equal. It is otherwise rather inefficient, while the converter is so devised as to be capable of maintaining acceptable efficiencies over some range not only of speed ratio but also of torque.

Fig. 10-19 indicates the general arrangement of a simple coupling in which the

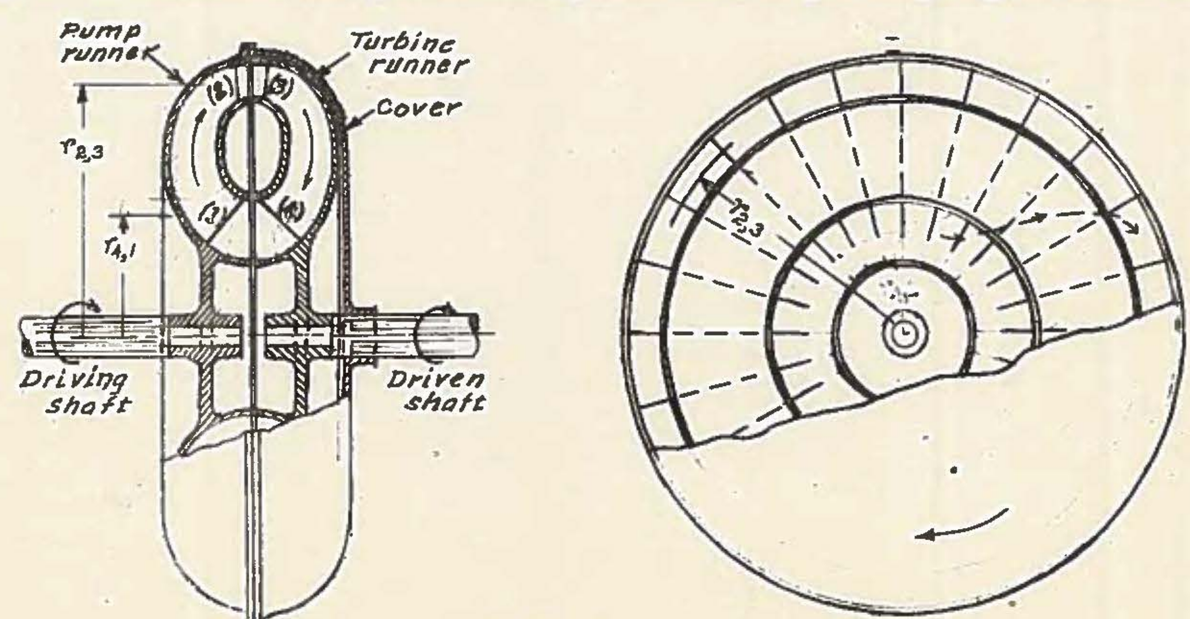


Fig. 10-19

fluid fills the runners and casing. The channels provided by the identical pump and turbine rotors are seen to have rather the appearance of a split doughnut, but with each divided into cells by radially oriented partitions corresponding to the curved blades of the conventional pump or turbine. This configuration restricts the efficient operating range of the unit, but permits operation in either direction or exchange in the functions of the two rotors. In the indicated arrangement the circulation produced by the turning of the pump rotor is from section (1) to section (2), and from (3) to (4) in the turbine rotor.

The powers supplied to the pump shaft and delivered by the turbine are expressible from the viewpoint of their torque and rotative speeds, or by energy accounting for the fluid circulating through and between the two. By the first,

$$P_p = T_p\,\omega_p; \quad \text{and} \quad P_t = T_t\,\omega_t$$

For the assembly of identical pump and turbine rotors the torques must be equal, else one would accelerate with respect to the other, so that $T_p = T_t = T_{p,t}$.

For the individual components of the assembly the energy relations are -

$$P_p = \dot{m}\left[(e_2 - e_1) + {}_1\phi_2\right], \text{ or } \dot{m}(\Delta e_p + {}_1\phi_2)$$

and

$$P_t = \dot{m}\left[(e_3 - e_4) - {}_3\phi_4\right], \text{ or } \dot{m}(\Delta e_t - {}_3\phi_4)$$

where e denotes the composite flow-work and kinetic energies at relevant locations and ϕ denotes the energy degenerating to internal energy in passage of the fluid through the rotors.

For the assembly account must also be made of energy dissipations occurring en route through the cross-over zones between the rotors. The aggregate so dissipated must be provided by an excess of Δe_p over Δe_t, or ${}_2\phi_3 + {}_4\phi_1 = \Delta e_p - \Delta e_t$. Any overall difference between the supplied and delivered powers is attributable to the aggregate of the energy dissipations, or

$$P_p - P_t = \dot{m}({}_1\phi_2 + {}_2\phi_3 + {}_3\phi_4 + {}_4\phi_1), \text{ or } \dot{m}\,\Sigma\phi,$$

and also, from above,

$$= T_{p,t}\,(\omega_p - \omega_t)$$

As more or less energy dissipation is inescapable, the power output by the turbine is likewise inescapably less than the input to the pump. Also, by the last relation, this will be reflected and measured by a deficit of the rotative speed of the output shaft relative to that of the input shaft. This is expressed as the slip of the coupling, and defined as

$$\text{Slip } (s) = (\omega_p - \omega_t)/\omega_p, \text{ or } 1 - \omega_t/\omega_p.$$

Or suitably regarding the ratio P_t/P_p as the transmission efficiency of the unit,

$$\text{Transmission efficiency } (\eta) = P_t/P_p \text{ and, from above,}$$
$$= \omega_t/\omega_p \text{ and thus, also,}$$
$$= 1 - \text{slip.} \qquad (10\text{-}27)$$

Considering the item of <u>slip</u> from an "exterior" viewpoint, it is in a sense necessitated by the imposing of a loading such that the transmitted torque is capable of turning the output shaft only at an appreciably less speed than that of the input shaft. The power output for which a coupling is "rated" is that at which the slip in the order of 3 to 5%, with a corresponding efficiency of 97 to 95%.

It is again convenient and informative to accept the approximation of representing by single vectors the stream velocities leaving or approaching the cells in the cross-over zones between the runners, with these taken further to be ones at mean radii $r_{2,3}$ and $r_{4,1}$.* The diagrams of fig. 10-20 relate such velocities as they would appear in a developed view of cylinders of such radii, in sketch (A) for a condition of moderate slip, but one of large slip in sketch (B). In each the <u>relative</u> velocities ($\bar{v}$) are shown as equal in the exit section of the cell of one runner and the entry section of the other, as $\bar{v} = \dot{m}/\rho A$ and the items to the right are equal. However, for reasons which are noted below, these differ in the two sketches.

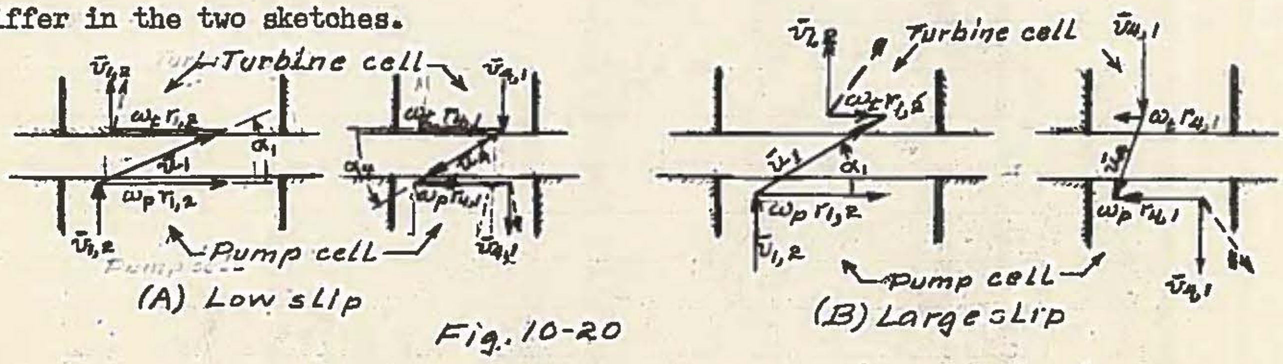


Fig. 10-20

In a manner comparable with the torque relation of eq. 10-2 for the centrifugal pump or a like one for the turbine, but noting that the orientation of the blades cause the angle α of the above figure to be 90° and its cotan thus zero, and that correspondingly $u \cos\alpha = \omega r$,

$$T_p, \text{ and/or } T_t, = \dot{m}\left[r_{2,3}\,(\bar{u}_2 \cos\alpha_2) - r_{4,1}\,(\bar{u}_4 \cos\alpha_4)\right]$$
$$= \dot{m}\left[\omega_p (r_{2,3})^2 - \omega_t (r_{4,1})^2\right] \qquad (10\text{-}28)$$

* - The character of the actual velocity distributions in the streams entering these zones is perhaps that of (a) the free vortex as regards radial distribution, and (b) a forced spiral character of vortex as regards the peripheral.

Note that for the favorable conditions of sketch (A) the angle α_4 at which the stream leaves the turbine runner is such as to permit rather undisturbed entry into the pump cells, but that for the condition of large slip in sketch (B) that angle is such as to cause active stream disturbance on such entry, with aggravated energy dissipation $({}_4\phi_1)$. Like observations apply with relation to the cross-over from pump to turbine. Both justify the active decrease in coupling efficiency which is found if the imposed output torque is such as to cause excessive slip. If this is maintained, and as the dissipated energy goes largely to increasing the fluid temperature, adequate cooling of the fluid must concurrently be provided.

For generalized correlation of the performance characteristics of geometrically similar couplings, dimensionless parameters which serve to advantage include the speed ratio (ω_t/ω_p) and/or the slip and the associated efficiency, and ones incorporating other variables such as the size, the rotative speed of the pump, and the torque transmitted or power delivered. Parameters including the latter are $T_{p,t}/(\rho d^5 \omega_p^2)$ or $P_p/(\rho d^5 \omega_p^3)$, and $P_t/(\rho d^5 \omega_p^3)$. Here **d** denotes a suitably representative diameter, such as the outer diameter of the rotors.

The attributing of specific magnitudes to these parameters implies that, for all of a family of geometrically similar couplings, the transmitted torque will be proportional to the fifth power of the rotor diameter if each were operated at a stated dynamic condition as regards slip and pump speed, or to the square of that speed for one of specified diameter; with a like proportionality of the deliverable power to the cube of the speed. These proportionalities may be anticipated by analyses and, to such extent as geometrical identities are obtainable in items such as channel contours and surface roughness for couplings of different sizes, are adequately validated by experience.

The curves of figure 10-21 are reasonably representative of the variation of the above parameters with the speed ratio or the slip for a family of typical and geometrically similar couplings. They are expressed, however, in terms only of their magnitudes relative to those at a rated operation at about 3 % slip. Observe here that, although the pump and turbine torques remain equal at all slips when the simple coupling is operating in equilibrium, even at a constant pump speed the transmittable torque increases with increase in the slip, to the point at which the output shaft is "stalled" by the restraining load, or at 100% slip.

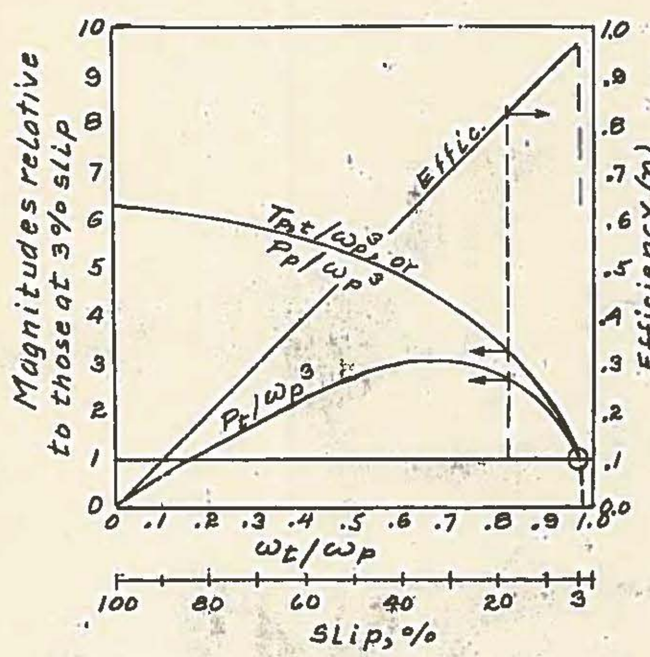


Fig. 10-20

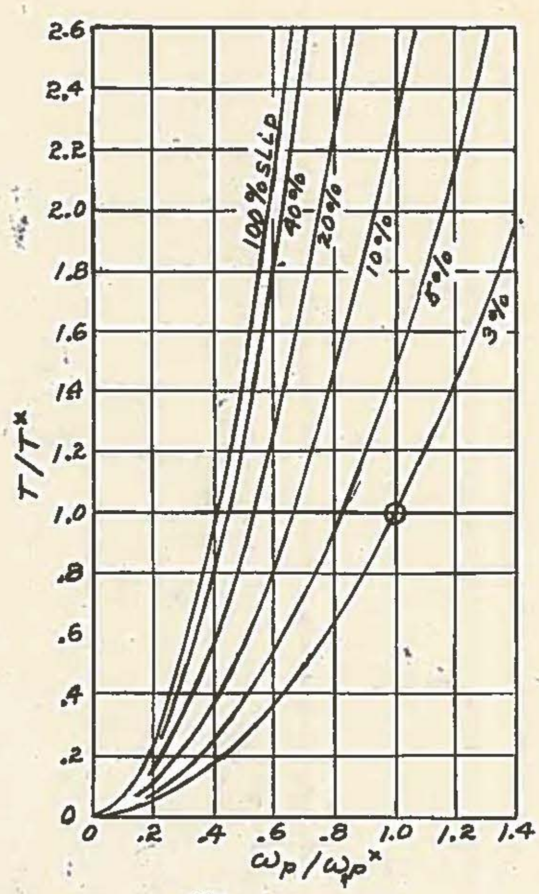


Fig. 10-22

Although it is accompanied by efficiency decrease, this torque increase with increase in slip is seen in the following to be in some instances a convenient characteristic of the coupling. The cause of this increase is that, as the pressure rise through the pump rotor and its decrease through the turbine are functions of the square of their respective rotative speeds, with decreasing magnitudes of ω_t/ω_p the increasing excess of Δp_p over Δp_t effects an increasing rate of fluid circulation ($\dot{m}$) until offset by a greater rate of energy dissipation per unit mass circulated.* A significant opposite consideration is that some excess of pump over turbine speed is required for effecting the fluid circulation, without which no torque may be transmitted. But the rapid approach to zero torque and power delivery at slips less than about 3% is attributable to a nullification of the small circulation stimulus by energy dissipation.

The curve indicating in fig. 10-20 the relative magnitudes of T/ω_p^2 (or of P_p/ω_p^3, both relative to those when the coupling is operating at 3% slip) is a semi-generalized equivalent of the family of contour lines reporting, in fig. 10-22, test results on a particular coupling when operated at a range of pump speeds up

(*- An estimate of the circulation rate may be made through eq. 10-28 if for a given coupling data are available concerning the relevant dimensions and speeds.)

to 140% of that for which the coupling is nominally rated, and when at each speed it is so loaded as to exhibit slips form 3% to 100%. The configuration of the contours may be expected to and normally does indicate a direct proportionality of T to ω_p^2. Paralleling fig. 19-21, a more fully generalized graph may evidently be developed through further descriptive data on the coupling size (d) and the density (ρ) of the working fluid.

The advantage of the characteristic torque increase with increase of the slip of the coupling is associated with the considerations that -

(a) the governing of the modern higher-speed internal combustion engine is typically of such nature that it may not attain full torque capability unless and until it is turning at a moderately high rotative speed; but

(b) the device which the engine is to actuate may well require both a rather high torque for starting it, and also some excess of torque for accelerating it to a normal operating speed compatible with that of the engine.

The automobile or truck imposes loading of this general character, as does other equipment. In such general situations the coupling may automatically permit the engine to gain a rotative speed sufficient to have provided the required starting torque, during which interval the slip is 100%, may then continue to furnish with decreasing slip the torques involved in accelerating, and if suitably sized may ultimately transmit with minor slip the torque required for operation at normal operating speed. In other characteristic situations in which for steady operation the required torque is itself about proportional to the square of the rotative speed (as with the marine propeller), the coupling may in normal operation maintain an optimum efficiency, but by increase of slip may provide the torque required for purposes of acceleration.

The following example illustrated various of the foregoing considerations.

Example 10-2. For a family of couplings having performance characteristics corresponding to those represented in figures 10-21 and 10-22, determine or estimate the following items for conditions as stated below.

(a) The numerical magnitudes of the generalized torque and power parameters, as evaluated using the subsequent (consistent) units, if one of 2.0 feet diameter is known to transmit a torque of 625 lbf.ft when operating at an input-shaft speed of 15 rev./sec with 3% slip and employing an oil with a density of 1.65 slugs/cu.ft.

(b) The like magnitudes at 10 and 100 % slip, and the torque and the power output at these slips and the same input-shaft speeds.

(c) The torque and the power output if, at the above slips, the input speed is 12 rps.

(d) The required rate of energy withdrawal as heat from the oil to maintain a constant temperature when operating at the 10% slip condition of part (b).

(e) The suitable coupling diameter for a loading which will normally require a torque of about 500 lbf.ft at an output-shaft speed of 900 rpm, but also the input-shaft speed required to provide a starting torque of about 2000 lbf.ft.

<u>Solution</u>. General items:- $d^5 = 32$; $\omega_p{}^2 =$ (a and b) 225 and (c) 144; $\omega_p{}^3 =$ (a and b) 3375 and (c) 1728; $\rho\, d^5\omega_p{}^2 =$ (a and b) 11,880 and (c) 7600; $\rho\, d^5\omega_p{}^3 =$ (a and b) 17.8x 10 and (c) 9.12x10 ; $P_p =$ (a) 2πx625x15 = 58,900 ft.lbf/sec (or 107 hp).

(a) $T/\rho\, d^5\omega_p{}^2 = 625/11,880 = 0.0526$ at 4% slip; $P_p/\rho\, d^5\omega_p{}^3 = 58,900/178,000 =$ 0.330 (or = 2πx torque parameter at the same slip).

(b) From the evidence of fig. 19-20, the torque parameters become about .0526x2.3 or 0.121 at 10% slip; and .0526 x 6.3 or .331 at 100% slip.

Computing from these the respective torques; T = 0.121x11,880 = 1440 lbf.ft at 10% slip, and .331x11,880 or 3930 at 100% slip. Or checking by fig. 10-22, at 10% slip and $\omega_p/\omega_p{}^* = 1.0$, $T = 2.3\ T^* = 2.3 \times 625 = 1440$.

Relative to their magnitudes at 3% slip, the power parameters will bear the same proportions as do the torque parameters, so that $P_p/\rho\, d^5\omega_p{}^3 = .330 \times 2.3 =$ .760 at 10% slip, and .330x6.3 or 2.08 at 100%.

From these the power outputs become; .330x178,000x(1-.03) or 57,000 ft.lbf/second at 3% slip, 0.760x178,000x(1-.10) of 122,000 at 10% slip, and 2.08x178,000 x(1-1) or zero at 100% slip. Or checking from the curve of power-output parameters of fig. 10-20, at 10% slip P_t = 2.1(.330x178,000) = 123,000 ft.lbf/sec.

(c) At 12 versus 15 rps the ratio of the torques will be in the proportion of $(12/15)^2$ or 0.64/1.0 to the above, and the powers in the proportion of $(12/15)^3$ or 0.512/1.0. Thus

Slip, %	3	10	100
T, lbf.ft	625x.64 = 400	1440x.64 = 922	3930x.64 = 2520
P_t, ft.lbf/sec	57,000x.512 = 29,300	122,000x.512 = 62,500	zero

Or checking by fig. 10-22, with $T/T^* = 1.47$ at $\omega_p/\omega_p{}^* = .8$ and slip = 10%, T = 1.47 x 625 = 920 (vs 922 above).

(d) Required rate of energy withdrawal = 0.10x(.760x178,000) or 13,550 ft.lbf per second (i.e., 13,550/778 or 17.4 Btu per second).

(e) Suitable diameter $= \left[\dfrac{500}{.0526 \times 1.65 \times (15/.97)^2}\right]^{1/5} = 26.2^{1/5} = 1.92$ ft.

At this diameter and for 100% slip, $\omega_p = \left[\dfrac{2000}{.331 \times 1.65 \times 26.2}\right]^{1/2} = 140^{1/2} = 11.8$ rev. per second, or 710 rpm.

It is evident from the foregoing that, when filled with the working fluid, the range of torque and power is quite limited at which a simple coupling of stated size will operate efficiently at a stated input speed. To give it a wider adaptability a common practice is that of providing for variation (while operating) of the amount of oil in the coupling. By thus reducing the transverse areas of the fluid streams traversing the cells of the rotors, and thereby the mass-rate of circulation, the associated torque may be reduced without excessive reduction of efficiency.

When for convenience a continued but low-speed turning of the input shaft is permitted, but it is desired that a minimum restraint be required for keeping the output shaft from turning, a suitably located obstruction to the fluid circulation may sufficiently lessen the otherwise "creeping" tendency with only minor influence on the efficiency at normal operating conditions.

10-11. Torque Converter. By suitably supplementing and modifying the fluid coupling, but making like wall-force adaptations, an arrangement known as the torque converter may be made capable of providing torque amplification with acceptable efficiency over some range of speed-reduction ratios (ω_t/ω_p). It is recalled that the simple coupling can do this only with marked sacrifice in efficiency. The general means for accomplishing this include -

(Turn to next page)

(a) the introducing of a stationary element, known as a reaction member or stator; and

(b) suitable orientation and configurations of the blading forming the cells of the rotors and stator.

These primary requirements may be supplemented by features such as the provision of more than one stage of turbine blading on the same rotor, or of an ability to revert to effectively the characteristics of the coupling when or if the desired torque and speed ratios approach unity. But for present purposes it is sufficient to give specific attention only to a simpler and pioneer (Foettinger) arrangement of converter such as is represented in fig. 10-23. In this the left half is the stator element, and the pump and turbine rotors occupy jointly the right half of the assembly. Accepting the approximations involved in representing by single vectors the relative and absolute velocities at the cross-over zones between the several elements, the several vector diagrams accompanying the figure serve to illustrate the relations between the absolute velocities (u) of the stream in the cross-overs and the relative velocities (v) of departure from and entry to the cells of the moving elements. The velocities and blading orientations are ones corresponding to a design and operation providing a torque multiplication of 2/1, at a speed ratio (ω_t/ω_p) of about 0.45 and associated (optimum) efficiency of about 90%.

Although giving an unrealistic impression of the actual blading configuration and velocity orientations (in space), fig. 10-24 represents a conventional and convenient manner of coordinating the relevant velocities at the successive cross-overs.

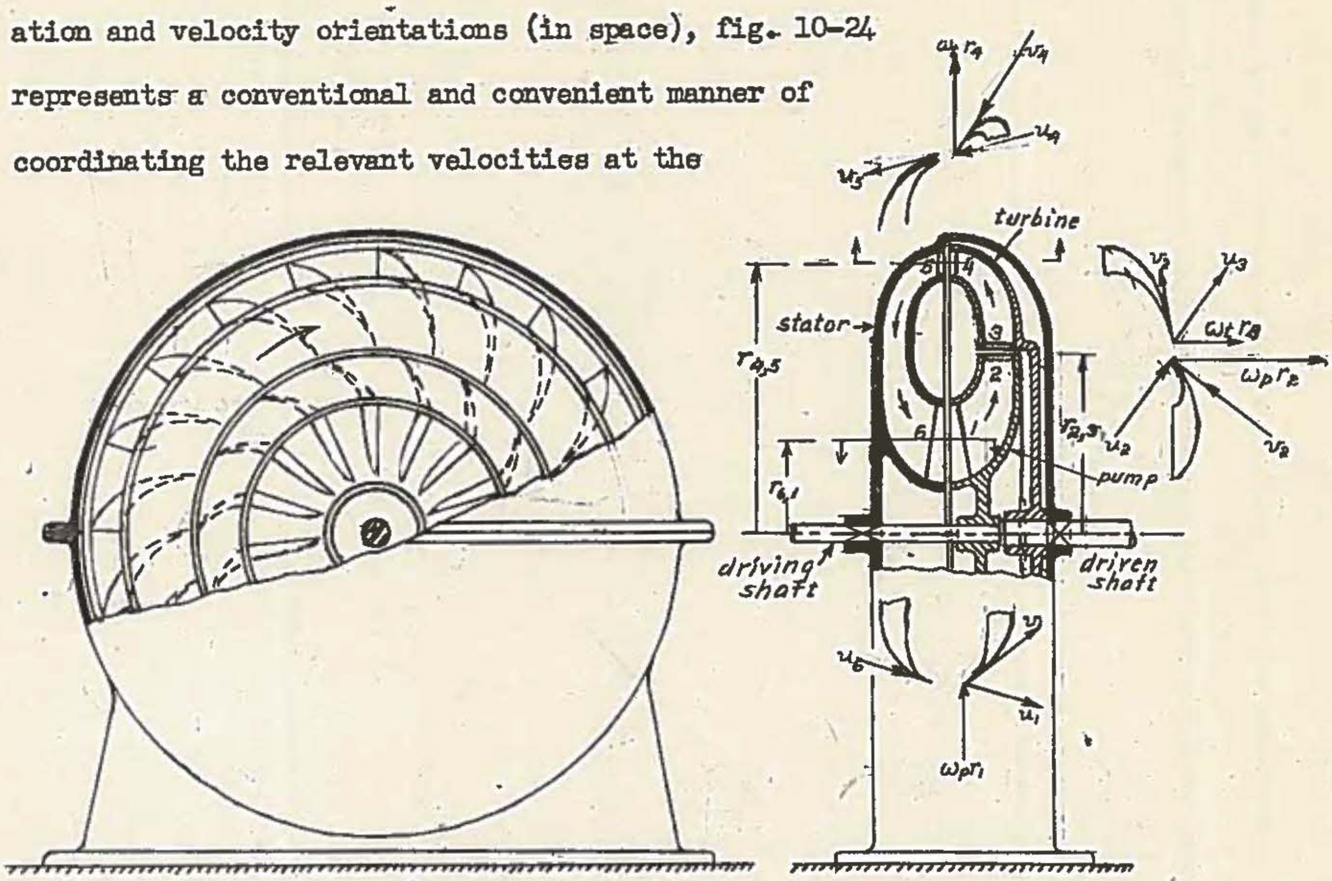


Fig. 10-23

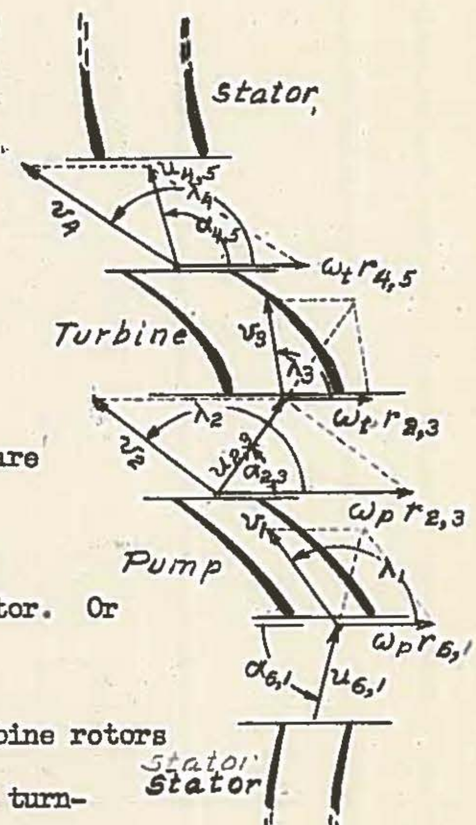


Fig. 10-24

The ability to provide torque multiplication efficiently is due to the features that -

(c) the orientation of the stator blading is such that the wall-force moment (torque) exerted on them by the circulating fluid acts in the same direction as that which through the same agency opposes the turning of the pump rotor, and the two are additative; and

(d) at equilibrial operation their sum is thus the torque available and acting to drive the turbine rotor. Or

$$T_s + T_p = - T_t$$

The negative sign denotes that, as the pump and turbine rotors turn in the same direction, the torque impeding the turning of the pump rotor is opposite in direction to that actuating the turbine rotor. With individual reference to these torques, and with symbol significances as indicated in fig. 10-24,

$$T_s/\dot{m} = r_6\, u_6 \cos\alpha_6 - r_{5,4}\, u_4 \cos\alpha_4 \;;$$

$$T_p/\dot{m} = r_2\, u_2 \cos\alpha_2 - r_{6,1}\, u_6 \cos\alpha_6 \;; \text{ and}$$

$$T_t/\dot{m} = -(r_{4,5}\, u_4 \cos\alpha_4 - r_{2,3}\, u_2 \cos\alpha_2)$$